International Journal of Engineering Research and Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 12, Issue 9 (September 2016), PP.25-31

Significant Deductions of Dynamic Systems in Vectorial Mechanics

Carlos Figueroa N.[1], Rafael Castillo O.[1], Gonzalo Moroyoqui E.[1]
[1] *Universidad de Sonora, Industrial Engineering Department.*

Abstract:- Important results of the vector mechanics are developed in an explicitly and educational form. The first calculation is a link for the algebraic process to obtain the acceleration in polar coordinates, once in curvilinear motion is usual to determine the radius of curvature, for example, in calculating the tangential acceleration; here the determined formula is explained. Finally, in the general model of collisions or impacts, the task is solving a system of simultaneous equations, which produces the conservation of energy and motion laws, which is useful also to understand the coefficient of restitution method. The three methods are commonly used in vector mechanics, however, almost always their elucidation is omitted. Our purpose is to provide in science and engineering, these significant deductions, in order to improve lessons and benefit to a better understanding and application of such models to real problems.

Keywords:- Kinetic energy, conservation laws, collisions, radius of curvature.

I. INTRODUCTION

In dynamic courses at the level of studying a major, the thematic programs have a great multiplicity of contents, the syllabus firstly include acceleration in Cartesian coordinates, likewise the definition of tangential and normal component; the speed and angular acceleration is the crucial issue, then is continued to express the acceleration in polar and cylindrical coordinates. Immediately, there are exercises which have been performed with applications of Newton's second law, moments, torques and peers. The rotational movement is significant since it is applicable to mechanisms and machines. Afterward, is continued with the elucidation of the work, energy, momentum and conservation laws. Then is studied the rigid body dynamics, airspeeds, articulated mechanisms and finally Dalembert principle. The amount of problems and exercises is of at least one thousand as is confirmed by subject texts such as the ones by Beer [1], Bedford [2] and Hibbeler [3].

The abundance of constructs leaves few time to develop and obtain certain deductions by elementary formulas of the course. The performance is strengthened if each item is accompanied by a demonstration of its main concepts, in that purport is necessary to develop some of them with application of algebra and differential calculus. Out of all justifications, we select some of interest, such as the development of acceleration in polar coordinates, the formula for the curvature radius, collisions, impacts and the restitution coefficient.

These issues are relevant in research, for example M.F. Ferreira Da Silva [4] studied impacts by using the geometric meaning of the restitution coefficient, another focus of interest is from Aníbal O. Garcia and Jose Pablo Cebreiro [5] who prove a dynamic model because of the variation of mass and movement that there is in an impact. B.F. Voronin and G. Villalobos H [6], determined the curvature radius of the cam on a flat mechanism. On the reference books side, Weisstein [6] and R. Tenenbaum [7] are useful in the more comprehensive understanding of the concepts.

Our contribution is didactic and responds to the excessive number of topics in the course program; therefore, is considered essential to generate this useful kind of materials in subjects related to science and engineering. The idea is that a work as the proposed here is innovative in synthesizing key derivations and leads to the design of auxiliary lessons in courses of dynamic, a material that helps effective study of the dynamics.

II. ACCELERATION IN POLAR COORDINATES

To represent vectors in polar coordinates, is defined a unit vector $\hat{e}_r$ pointing in the radial direction, so that the position where is the vector is determined from O to P

$$\vec{r} = r\hat{e}_r \quad (1)$$

The velocity is obtained deriving in relation to the time. (1)

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} \qquad (2)$$

When P moves across a curvilinear path, the unit vector $\hat{e}_r$ rotates with angular velocity

$$\omega = \frac{d\theta}{dt} \quad (3)$$

The derivative in relation to $\hat{e}_r$ time may be expressed in terms of $\hat{e}_\theta$ and it can be stated as

$$\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}\hat{e}_\theta \qquad (4)$$

Substituting this in (2), is obtained the velocity of point P, also is possible to replace the angular velocity (3)

$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta = \frac{dr}{dt}\hat{e}_r + r\omega\hat{e}_\theta \qquad (5)$$

To demonstrate the equation (4), and according to figure 1, is obtained:

$$\hat{e}_r = \cos\theta\hat{i} + sen\theta\hat{j} \qquad (6)$$

$$\hat{e}_\theta = -sen\theta\hat{i} + \cos\theta\hat{j} \qquad (7)$$

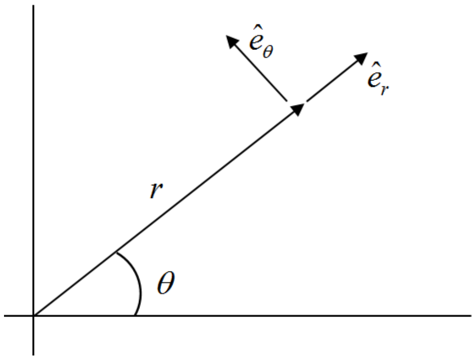

Fig. 1. Unit vectors

The relation between $\hat{e}_r$ and $\hat{e}_\theta$ is that they are perpendicular $\hat{e}_r \cdot \hat{e}_\theta = 0$, so it is only about to derive the relation (6)

$$\frac{d}{dt}\hat{e}_r = \frac{d}{dt}(\cos\theta_i + sen\theta_j) \, (8)$$

$$\frac{d\hat{e}_r}{dt} = \left(-sen\theta_i\frac{d\theta}{dt} + \cos\theta_j\frac{d\theta}{dt}\right) \qquad (9)$$

$$\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}\left(-sen\theta_i + \cos\theta_j\right) \quad (10)$$

Where is identified the unit vector (7)

$$\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}\hat{e}_\theta \qquad (11)$$

Desired result.

The acceleration is obtained by deriving the equation (5) in relation to the time

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\right] \qquad (12)$$

$$\frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta + r\frac{d\theta}{dt}\frac{d\hat{e}_\theta}{dt} \qquad (13)$$

In the last term can be stated that

$$\frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r \qquad (14)$$

In effect at deriving the relation (7)

$$\frac{d}{dt}\hat{e}_\theta = \frac{d}{dt}(-sen\theta_i + \cos\theta_j) \quad (15)$$

At deriving in relation to time

$$\frac{d\hat{e}_\theta}{dt} = \left(-\cos\theta_i\frac{d\theta}{dt} - \operatorname{sen}\theta_j\frac{d\theta}{dt}\right) \qquad (16)$$

$$\frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\left(\cos\theta_i + \operatorname{sen}\theta_j\right) \quad (17)$$

$$\frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r \qquad (18)$$

The previous result is replaced in the acceleration vector (13) to generate the result in polar coordinates

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{e}_\theta \quad (19)$$

III. CURVATURE RADIUS

The second formula to deduce is the curvature radius, commonly used in dynamics, it is known that for a curve $y = f(x)$, the curvature radius is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \qquad (20)$$

For obtaining the aforementioned formula are used parametric equations given by

$$x = x(t)$$
$$y = y(t) \qquad (21)$$

In these equations terms, the curvature radius is given by

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} \qquad (22)$$

Figure 2 describes the geometrical interpretation of a radius of any curve.

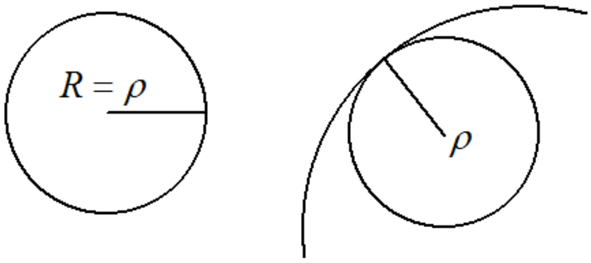

Fig. 2. Radius of a curve

In the next figure 3 is described a $y = f(x)$ curve

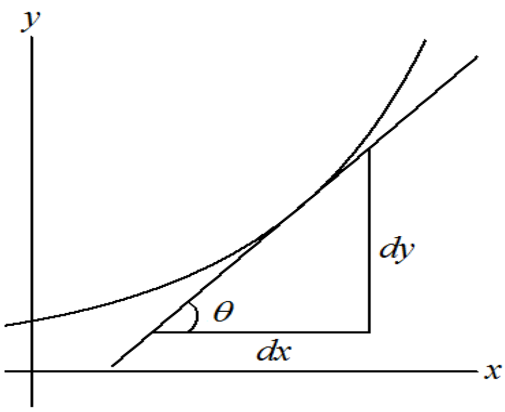

Fig. 3. Tangent line.

Also in Figure 3 is observed that for any curve

$$\tan\theta = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad (23)$$

Remembering its respective derivate

$$\frac{d\tan\theta}{d\theta} = \sec^2\theta\dot{\theta} \quad (24)$$

Equating both derivatives has

$$\sec^2\theta\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \quad (25)$$

Using the trigonometric identity

$$1 + \tan^2\theta = \sec^2\theta \quad (26)$$

It now has

$$\left[1 + \tan^2\theta\right]\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \quad (27)$$

Which can be rewritten as

$$\left[1 + \frac{\dot{y}^2}{\dot{x}^2}\right]\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \quad (28)$$

likewise

$$\left[\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2}\right]\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \quad (29)$$

$$\left[\dot{x}^2 + \dot{y}^2\right]\dot{\theta} = \dot{x}\ddot{y} - \ddot{x}\dot{y} \quad (30)$$

Is determined

$$\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}, \quad (31)$$

Furthermore, a differential curve is given by the Pythagorean theorem. According to Figure 4 it is

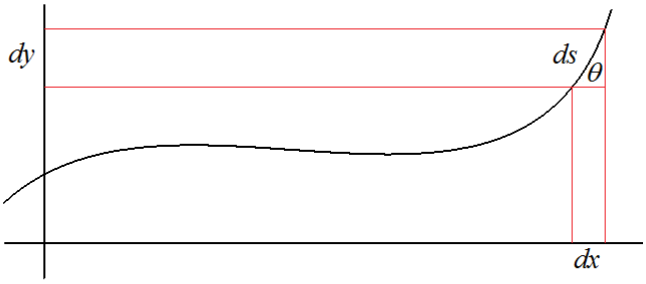

Fig. 4. length of arc.

$$ds^2 = dx^2 + dy^2 \quad (32)$$

$$\dot{s}^2 = \sqrt{\dot{x}^2 + \dot{y}^2} \quad (33)$$

In polar coordinates, a differential arc according to Figure 5, is given by

$$ds = \rho d\theta \quad (34)$$

$$\rho = \frac{ds}{d\theta} = \frac{\dot{s}}{\dot{\theta}} \quad (35)$$

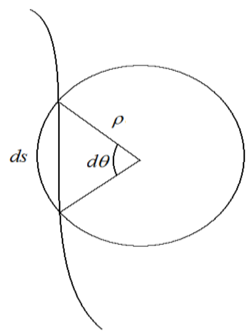

Figure 5. Length of arc

Substituting (31) and (33) both equations in (35) is determined that

$$\rho = \frac{\left[\dot{x}^2 + \dot{y}^2\right]^{\frac{1}{2}}}{\dfrac{1}{\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}}} \qquad (36)$$

With algebra is generated the curvature radius for parametric equations, equation (22)

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} \qquad (37)$$

The formula for a $f(x)$ is given by the second derivative and making explicit the derivative in relation to time [8].

$$\frac{dy^2}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) \qquad (38)$$

$$= \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)\left(\frac{dt}{dt}\right) \qquad (39)$$

$$= \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)\left(\frac{1}{dt}\right)\left(\frac{1}{\dot{x}}\right) \qquad (40)$$

In fact, if it has that $\dfrac{dx}{dt} = \dot{x}$, equation (40) it is correct.

Deriving the formula quotient

$$\frac{d^2y}{dx^2} = \left(\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2}\right)\left(\frac{1}{\dot{x}}\right) \qquad (41)$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} \qquad (42)$$

Now from the parametric variables formula for equation (37) is done algebra, so that the second derivative is identified

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} = \frac{\dfrac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\left(\dot{x}\right)^3}}{\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\dot{x}\right)^3}} = \frac{\dfrac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\left(\dot{x}^2\right)^{3/2}}}{\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\dot{x}\right)^3}} \qquad (43)$$

The curvature radius is obtained

$$\rho = \frac{\left[1 + \left(\dfrac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2}}{\left(\dfrac{d^2y}{dx^2}\right)} \qquad (44)$$

IV. COLLISIONS AND RESTITUTION COEFFICIENT

Then, is presented the third instance to demonstrate on the theme of impacts, it is started generating the formula for two masses in motion before the collision, afterwards assume a collision with different initial velocities and considering primed velocities after the collision, the conservation of kinetic energy and moment is

$$m_1 v_1^{\,2} + m_2 v_2^{\,2} = m_1 v'^{\,2}_1 + m_2 v'^{\,2}_2 \qquad (45)$$

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \qquad (46)$$

Rewriting the equation (46) as

$$m_1 v_1 - m_1 v'_1 = m_2 v'_2 - m_2 v_2 \qquad (47)$$

$$m_1\left(v_1 - v'_1\right) = m_2\left(v'_2 - v_2\right) \qquad (48)$$

Similarly, the energy conservation is rewritten, equation (45)

$$m_1 v_1^2 - m_1 v'^2_1 = m_2 v'^2_2 - m_2 v_2^2 \qquad (49)$$

$$m_1\left(v_1^2 - v'^2_1\right) = m_2\left(v'^2_2 - v_2^2\right) \qquad (50)$$

Can be expressed as

$$m_1\left(v_1 - v'_1\right)\left(v_1 + v'_1\right) = m_2\left(v'_2 - v_2\right)\left(v'_2 + v_2\right) \qquad (51)$$

Herein identified in equation (51) the expression (48) result then

$$v_1 + v'_1 = v'_2 + v_2 \qquad (52)$$

$$v'_1 = v'_2 + v_2 - v_1 \qquad (53)$$

This result for equation (53) substitute in (46) for v'_1

$$m_1 v_1 + m_2 v_2 = m_1\left(v'_2 + v_2 - v_1\right) + m_2 v'_2 \qquad (54)$$

$$m_1 v_1 + m_2 v_2 = m_1 v'_2 + m_1 v_2 - m_1 v_1 + m_2 v'_2 \quad (55)$$

$$m_1 v_1 + m_2 v_2 - m_1 v_2 + m_1 v_1 = v'_2\left(m_1 + m_2\right) \qquad (56)$$

The mass 2 velocity after the collision is

$$v'_2 = \frac{2m_1 v_1 + m_2 v_2 - m_1 v_2}{m_1 + m_2} \qquad (57)$$

$$v'_2 = \frac{2m_1 v_1}{m_1 + m_2} + \frac{m_2 - m_1}{m_1 + m_2} v_2 \qquad (58)$$

If mass 2 is in repose before the collision, the equation (58) becomes

$$v'_2 = \frac{2m_1 v_1}{m_1 + m_2} \quad (59)$$

Furthermore, in the same manner for the mass one is proceeded as same at solving for (52) has

$$v'_2 = v'_1 + v_1 - v_2 \qquad (60)$$

By substituting this expression in equation (46) is obtained

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2\left(v'_1 + v_1 - v_2\right) \qquad (61)$$

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_1 + m_2 v_1 - m_2 v_2 \qquad (62)$$

$$m_1 v_1 + m_2 v_2 - m_2 v_1 + m_2 v_2 = v'_1\left(m_1 + m_2\right) \qquad (63)$$

$$v'_1 = \frac{2m_2 v_2 + v_1\left(m_1 - m_2\right)}{m_1 + m_2} \qquad (64)$$

It has the velocity of mass one after the collision

$$v'_1 = 2\frac{m_2 v_2}{m_1 + m_2} + \frac{\left(m_1 - m_2\right)}{m_1 + m_2} v_1 \quad (65)$$

If mass one is initially in repose, it has

$$v'_1 = 2\frac{m_2 v_2}{m_1 + m_2} \qquad (66)$$

Finally, is obtained the formula for the coefficient of restitution for an ideal case (perfectly elastic) for two masses m_1 and m_2 with velocities v_{ai} before impact and v_{bi}, after ; applying energy conservation

$$\frac{1}{2}m_1 v_{a1}^2 + \frac{1}{2}m_2 v_{a2}^2 = \frac{1}{2}m_1 v_{b1}^2 + \frac{1}{2}m_2 v_{b2}^2 \qquad (67)$$

$$m_1 v_{a1}^2 + m_2 v_{a2}^2 = m_1 v_{b1}^2 + m_2 v_{b2}^2 \quad (68)$$

$$m_1 v_{a1}^2 - m_1 v_{b1}^2 = m_2 v_{b2}^2 - m_2 v_{a2}^2 \quad (69)$$

$$m_1\left(v_{a1}^2 - v_{b1}^2\right) = m_2\left(v_{b2}^2 - v_{a2}^2\right) \quad (70)$$

Also recurring to the momentum conservation

$$m_1 v_{a1} + m_2 v_{a2} = m_1 v_{b1} + m_2 v_{b2} \quad (71)$$

$$m_1 v_{a1} - m_1 v_{b1} = m_2 v_{b2} - m_2 v_{a2} \quad (72)$$

$$m_1 \left(v_{a1} - v_{b1}\right) = m_2 \left(v_{b2} - v_{a2}\right) \quad (73)$$

Dividing the equation (70) by (73) is determined

$$\frac{m_1\left(v_{a1}{}^2 - v_{b1}{}^2\right)}{m_1\left(v_{a1} - v_{b1}\right)} = \frac{m_2\left(v_{b2}{}^2 - v_{a2}{}^2\right)}{m_2\left(v_{b2} - v_{a2}\right)} \quad (74)$$

$$\frac{\left(v_{a1}{}^2 - v_{b1}{}^2\right)}{\left(v_{a1} - v_{b1}\right)} = \frac{\left(v_{b2}{}^2 - v_{a2}{}^2\right)}{\left(v_{b2} - v_{a2}\right)} \quad (75)$$

By developing the difference of squares

$$\frac{\left(v_{a1} - v_{b1}\right)\left(v_{a1} + v_{b1}\right)}{\left(v_{a1} - v_{b1}\right)} = \frac{\left(v_{b2} - v_{a2}\right)\left(v_{b2} + v_{a2}\right)}{\left(v_{b2} - v_{a2}\right)} \quad (76)$$

Is obtained

$$v_{a1} + v_{b1} = v_{b2} + v_{a2} \quad (77)$$

$$v_{b1} - v_{b2} = v_{a2} - v_{a1} \quad (78)$$

Then the restitution coefficient (e) is defined as the negative ratio of the relative velocity after the collision between the relative velocity before the collision

$$e = -\frac{v_{b1} - v_{b2}}{v_{a1} - v_{a2}} \quad (79)$$

By eliminating the negative sign, it has:

$$e = \frac{v_{b2} - v_{b1}}{v_{a1} - v_{a2}} \quad (80)$$

The value of (e) shall be between 0 and 1, being totally elastic in 1 and completely inelastic in 0. The result is related to the conservation of energy

V. RESULTS DISCUSSIONS

In the acceleration in polar coordinates are obtained equations (4) and (11) provided in the respective courses resolution as obvious, also is a crucial link in the process of setting the acceleration vector. In the curvature radius is a calculation corresponding to a differential kind, is about joining steps in simplified form where key combinations are in equations (38) and (40). The point quid in the development of the velocities equation after impact is in (50) and (52) which always creates difficulty for students when solving, finally, the restitution coefficient is another alternative method by integral, here it is used the conservation laws in line with the velocities method seen here. It is noticed that with minimal arguments can be explained important concepts of body dynamics.

VI. CONCLUSION

This type of work can be valuable to help fully understand the concepts of vector dynamics. The first conclusion is that the relation with differential and integral calculus is significant; for example, the acceleration in polar coordinates is always a problem in a learning process, this perhaps because there is no solid basis to know the root of the model. The curvature radius is a study of flat curves, it is necessary to know tangential and normal lines to a curve for a more precise deduction. Such deduction should be studied in detail in the preceding dynamic courses, although the solution provided here is enough justification. The second conclusion is the issue of collisions and restitution coefficient, closely related to the conservation laws of energy and motion, but requires minimal skills in linear algebra, this other course that serve as the base to the dynamics. Finally, we must mention other deductions that must effectuate in the course, such as the force in terms of the derivative of a potential, calculation of moments of inertia, Newton's second law with the conservation movement, Euler angles, principle Dalembert. Everything can be material to design lessons of the type here proposed.

REFERENCES

[1]. F. Beer y E.R. Johnston, (1998) Mecánica Vectorial para Ingeniería, Dinámica, sexta edición, Edit. McGraw Hill, México, pp. 794-796.
[2]. A. Bedford y W. Fowler, (2008) Mecánica para Ingeniería, Dinámica, quinta edición, Edit. Pearson Prentice Hall, México, pp. 68-70.
[3]. R.C. Hibbeler, (2010) Ingeniería Mecánica, decima segunda edición, Prentice Hall, pp. 68-70, México.
[4]. M.F. Ferreira Da Silva, (2008) Revista Mexicana de Física, 54 (1), pp. 65-74.
[5]. A. O. García y J.P. Cebreiro, (2008) Modelo Dinámico de colisión, Congreso Iberoamericano de Accidentología Vial, Argentina.
[6]. B.F. Voronin y G. Villalobos H. (2010) Determinación del radio de la curvatura de la leva de un mecanismo plano, Revista Iberoamericana de Ingeniería Mecánica, 14 1, pp. 15-25.
[7]. E. W. Weisstein, Edit. Mathworld Wolfram Research
[8]. K. McAloon y A. Tromba, (1976) Cálculo, primera edición, Edit. PCSA, México, pp. 258-259.