International Journal Of Engineering Research And Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 13, Issue 11 (November 2017), PP.62-67

Bounded Composition Operators on Hardy Spaces

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ABSTRACT: We show analytic functions $\varphi, (\varphi+\varepsilon)$ on $\mathbb{R}$, such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$. We can define a weighted composition operator by $f \longmapsto (\varphi+\varepsilon)(f\circ\varphi)$. In this work we deal with the boundedness, compactness, weak compactness, and complete continuity of weighted composition operators on Hardy spaces $H_{\delta+1}(\delta > 0)$. In particular, we prove that such an operator is compact on H_0 if and only if it is weakly compact on this space. This result depends on a technique which passes the weak compactness from an operator T_{r-1} to operators dominated in norm by $\mathbb{T}$.

Keywords: weighted composition operators; Hardy spaces; compact operators.

Date of Submission: 24-10-2017 Date of acceptance: 02-11-2017

I. INTRODUCTION

When does a weighted composition operator map the Hardy space $H_{\delta+1}$ into itself? A weighted composition operator $W_{\varphi,(\varphi+\varepsilon)}$ is an operator that map $s \in \mathcal{K}(\mathbb{R})$, the space of holomorphic functions on the unit disk $\mathbb{R}$, into $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}(f)(z) = (\varphi+\varepsilon)(z)f(\varphi(z))$, where φ and $(\varphi+\varepsilon)$ are analytic functions defined in $\mathbb{R}$ such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$. These operators turn up in a natural way. And Forelli obtained the same result for the Hardy spaces $H_{\delta+1}$ when $\delta > 0$,$\delta+1 \neq 2$(see[7,9]).

When $(\varphi+\varepsilon) \equiv 1$, we just have the composition operator C_φ defined by $C_\varphi(f) = f\circ\varphi$. In this case, Littlewood's subordination theorem says that $C_\varphi(f) \epsilon H_{\delta+1}$ whenever $f \epsilon H_{\delta+1}$; that is, $C_\varphi: H_{\delta+1} \longrightarrow H_{\delta+1}$ is a continuous linear map for $\delta \geq 0$ [3]. The situation is really different when we consider weighted composition operators $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ on $H_{\delta+1}$. It is easy to find examples where $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}(H_{\delta+1}) \nsubseteq H_{\delta+1}$. In part II, we characterize the boundedness of $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ from $H_{\delta+1}$ into $H_{\delta+1}$.

In part III, we tackle this problem from three points of view: we study the cases where $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ is compact, weakly compact, and completely continuous on $H_{\delta+1}$. Let us recall that an operator T_{r-1} from a Banach space X into another Banach space Y is said to be compact if T_{r-1} maps bounded subsets into relatively norm compact sets; T_{r-1} is said to be weakly compact if it maps bounded subsets into relatively weakly compact sets; and T_{r-1} is said to be completely continuous if it maps weakly compact subsets into compact sets. It is well known that if T_{r-1} is compact, then it is weakly compact and completely continuous. The other implications are not true in general.

In [13], Proved that if a composition operator is weakly compact on H_0, then it is, in fact, a compact operator on this space. In Theorem 3.4, we prove that $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$is compact on H_1if and only if it is weakly compact on this space. And a result which passes the weak compactness from one operator T_{r-1} to another operator S_{r-1} such that $\|S_{r-1}x\| \leq \|T_{r-1}x\|$ for all x see Proposition 3.2.A similar result for compact operators is well known and can be seen, for example, in [5]. We think that our proof is more elementary. It is worth mentioning that in [2,15] the first named author and D´ıaz-Madrigal proved that $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$is compact on H_∞if and only if it is weakly compact on it.

In Section III, we also characterize the case where $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ is compact on H_∞; $\delta > 0$. Note that when $\delta < 0 < \infty$, $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ is always weakly compact on $H_{\delta+1}$. Although the classes of completely continuous and compact weighted composition operators agree on $H_{\delta+1}$ for $\delta > 0$ this result is obvious for $\delta > 0$, and it can be seen in [2] for $\varepsilon = \infty$, they are not the same on H_0 . This was pointed out for composition operators by Cima and Matheson [1,15] by showing that the composition operator C_φ, with $\varphi(z) = z\left(\frac{z+1}{2}\right)$, is completely continuous on H_0, but it is not compact. In part 4,we study the case where $(W_{r-1})_{\varphi,(\varphi+\varepsilon)}$ is completely continuous on H_0. For composition operators this result was obtained by [1,15].

In what follows we denote by $\mathbb{T}$the unit circle, by m the normalized
Lebesgue measure on , and by $\|f\|_{\delta+1}$the usual norm of a function $f \in H_{\delta+1}$. We refer the reader to [3,9,14,15] for the terminology and results about spaces of analytic functions.

II. BOUNDEDNESS

In this part we characterize the boundedness of $(W_{r-1})_{\varphi,\delta+1}$on $H_{\delta+1}$ in terms of a Carleson measure criterion. This criterion has been used to characterize boundedness of composition operators (see, [10,11,15]).

Definition 2.1. A positive measure μ on $\overline{\mathbb{R}}$ is called a Carleson measure ($\mathbb{R}$). if there is a constant $M < \infty$ such that $\mu(S_{r-1}(b,\tau)) \leq Mr$ forall $b \in \mathbb{T}$ and $\tau < 0$, where $S_{r-1}(b,\tau) = \{z \in \overline{\mathbb{R}}: |z-b| \leq \tau\}$.

Most of the information we are going to obtain about weighted composition operators will be given in terms of a certain measure, which we turn to next. Given an analytic function φ of the unit disk into itself, it is well known from Fatou's theorem that the radial limits $\lim_{r\to 1} -\varphi(re^{i\theta})$. exist almost everywhere. So, we can consider φ as a function belonging to $L_\infty(\mathbb{T}, m)$. Thus, taking $H_{\delta+1}$, we can define the measure $\mu_{\varphi,(\varphi+\varepsilon),(\delta+1)}$on $\overline{\mathbb{R}}$ by

$$\mu_{\varphi,\varphi+\varepsilon,\delta+1}(E) := \int_{\varphi^{-1}(E)<\cap \mathbb{T}} |\varphi+\varepsilon|\, dm,$$

Where E is a measurable subset of the unit closed disk $\overline{\mathbb{R}}$.

The next lemma will be crucial in what follows. In fact, it is a slight generalization of [8,15].

Lemma 2.1. From [8]. Fixing $\delta \geq 0$ and given $\varphi, \varphi+\varepsilon \in H_{\delta+1}$ such that $\varphi(\mathbb{R})$ we have

$$\int_{\overline{\mathbb{R}}} g d\, \mu_{\varphi,\varphi+\varepsilon,\delta+1} = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1}(go\varphi)dm$$

where g is an arbitrary measurable positive function in $\overline{\mathbb{R}}$.

Proof.If g is a measurable simple function defined on $\overline{\mathbb{R}}$ given by $g = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, we have that

$$\int_{\overline{\mathbb{R}}} gd\mu_{\varphi,\varphi+\varepsilon,\delta+1} = \sum_{i=1}^{n} \alpha_i\, \mu_{\varphi,\varphi+\varepsilon,\delta+1}(E_i)$$

$$= \sum_{i=1}^{n} \alpha_i \int_{\varphi^{-1}(E)\cap\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} dm = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1}\left(\sum_{i=1}^{n} \alpha_i\, \chi_{\varphi^{-1}} \quad (E_i)\cap\mathbb{T}\right)dm = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} go\varphi dm$$

Now, if g is a measurable positive function in $\overline{\mathbb{R}}$, we take an increasing
Sequence (g_n). of positive and simple functions such that $(g_n(z)) \to g(z)$. For all $z \in \overline{\mathbb{R}}$. Then, we have $\int_{\overline{\mathbb{R}}} g_n d\mu_{\varphi,\varphi+\varepsilon,\delta+1}$. On the other hand, $(|\varphi+\varepsilon|^{\delta+1} g_n o\varphi)$ is an increasing sequence such that $\Big(|\varphi +$ εδ+1zgnφz→φ+εδ+1zgφz for all zϵ ℝ, so

$$\int_{\overline{\mathbb{R}}} g_n d\mu_{\varphi,\varphi+\varepsilon,\delta+1} = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} g_n o\varphi\, dm \to \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} go\varphi\, dm$$

An obvious necessary condition for $W_{\varphi,\varphi+\varepsilon}W$ to be bounded on $H_{\delta+1}$ is that $\varphi+\varepsilon = (W_{r-1})_{\varphi,\varphi+\varepsilon}(1) \in H_{\delta+1}$. Whereas this condition is trivially sufficient for $\delta = \infty$ it is not sufficient for $\delta < \infty$.

Theorem 2.2. Fixing $\delta \geq 0$ and given $(\varphi, \varphi+\varepsilon) \in H_{\delta+1}$ such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$, we have that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is bounded on $H_{\delta+1}$ if and only if $\mu_{\varphi,\varphi+\varepsilon,\delta+1}$ is a Carleson measure in $\overline{\mathbb{R}}$.

Proof. On the one hand, by [3] ,$\mu_{\varphi,\varphi+\varepsilon,\delta+1}$is a Carleson measure in $\overline{\mathbb{R}}$ if and only if there is a constant $C > 0$ so that

$$\int_{\overline{\mathbb{R}}} |f|^{\delta+1} d\mu_{\varphi,\varphi+\varepsilon,\delta+1} \leq C\|F\|_{\delta+1}^{\delta+1}$$

for all $f \in H_{\delta+1}$. On the other hand, by Lemma 2.1, taking $g = |f|^{\delta+1}$, we have that

$$\int_{\overline{\mathbb{R}}} |f|^{\delta+1} d\mu_{\varphi,\varphi+\varepsilon,\delta+1} = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} |fo\varphi|^{\delta+1} dm = \left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}\right\|_{\delta+1}^{\delta+1}$$

Hence, $\mu_{\varphi,\varphi+\varepsilon,\delta+1}$is a Carleson measure in $\overline{\mathbb{R}}$ if and only if there is a constant $C > 0$ so that $\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(f)\right\|_{\delta+1} \leq C^{1/\delta+1}\|f\|_{\delta+1}$for all $f \in H_{\delta+1}$.

In[12,15] , got a sufficient condition of the boundedness of $(W_{r-1})_{\varphi,\varphi+\varepsilon}$on H_1. Namely, they showed that if the measures given by

$$\mu(E) := \int_E |(\varphi+\varepsilon)'(z)|^2(1-|z|^2)\, dA(z)$$

and

$$v(E) := \int_E |\varphi+\varepsilon(z)|^2|(\varphi+\varepsilon)'(z)|^2(1-|z|^2)\, dA(z)$$

for every measurable subset E of $\overline{\mathbb{R}}$, where A denotes the Lebesgue measure on $\overline{\mathbb{R}}$, satisfy

$$\sup_{0<\tau<1, b\epsilon\mathbb{T}} \frac{\mu(S_{r-1}(b,\tau))}{\tau^3} < \infty \quad \text{and} \quad \sup_{0<\tau<1, b\epsilon\mathbb{T}} \frac{v(S_{r-1}(b,\tau))}{\tau^3} < \infty$$

Then $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is bounded on H_2

III. COMPACTNESS AND WEAK COMPACTNESS

In this part, we present the main result of this work, namely, every weakly compact weighted composition operator on H_0 is compact on this space. Its proof leans on the following preliminary results. The first one can be found in [4,15].

Lemma 3.1. Let(x_n). be a bounded sequence in a Banach space X. Then (x_n). Is weakly null if and only if for each subsequence (x_{n_k}). there is a k sequence of convex combinations of(x_{n_k})., that we denote by (y_n), such that k$\|y_n\| \to 0$.

Proposition 3.2.Let X, Y, and Z be Banach spaces, and let $T_{r-1}: X \to Y$
and $S_{r-1}: X \to Z$ be bounded operators such that $\|S_{r-1}x\| \leq \|T_{r-1}x\|$ for all $x \in X$.
Suppose that there are two linear topologies τ_1 on X and $\tau_2$on Y such that T_{r-1} is $\tau_1 - \tau_2$continuous,(B_X, τ_1).ismetrizable and compact, and the weak topology of Y is finer than τ_2. If T_{r-1} is weakly compact, then so is S_{r-1}.
Before proving this proposition, it is worth mentioning that we plan to
apply it to the spaces $X = Y = H_1, \tau_1$ the topology of uniform convergence on compact sets,τ_2 the topology of the pointwise convergence, and $T_{r-1} = (W_{r-1})_{\varphi,\varphi+\varepsilon}$.
Proof. Let(x_n). be a sequence in B_X. We have to find a subsequence (x_{n_k}). Of (x_n). such that (Sx_{n_k}). converges in the weak topology of Z.
Since (B_X, τ_1). Ismetrizable and compact, there is a subsequence(x_{n_k}). of (x_n). and a point $d\ x\epsilon B_X$such that $(x_{n_k} - x)$. converges to zero in the topology τ_1. This is the subsequence we are looking for. Now, using Lemma 3.1, we are going to prove that $\left(S_{r-1}(x_{n_k} - x)\right)$ is a weakly null k sequence. Bearing in mind that T_{r-1} is $\tau_1 - \tau_2$ continuous, the weak topology of Y is finer than τ_2 , and T_{r-1} is weakly compact, we have that $\left(T_{r-1}(x_{n_k} - x)\right)$converges to zero in the weak topology. Let us take a subsequence Y_k. of (x_{n_k}). Then there is a sequence(z_k). of convex combinations of the Y_k such that $\|S_{r-1}(z_k - x)\| \to 0$. Since $\|S_{r-1}(z_k - x)\| \leq \|T_{r-1}(z_k - x)\|$, we have that $\|S_{r-1}(z_k - x)\| \to 0$. Summing up, for each subsequence (y_k) of (x_{n_k}), we have k found a sequence (z_k). of convex combinations of the y_k such that $\|S_{r-1}(z_k - x)\| \to 0$. By Lemma 3.1, $\left(S_{r-1}(x_{n_k} - x)\right)$ converges to zero in the k weak topology.
The proof of the following lemma can be obtained by adapting the proof
of [3].
Lemma 3.3. For $\delta \geq 0$ and $\varphi, (\varphi+\varepsilon)\epsilon H_{\delta+1}$ such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$ and $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is continuous on $H_{\delta+1}$, we have that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is compact on $H_{\delta+1}$ if and only whenever f_n is bounded on $H_{\delta+1}$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{R}$, then $\left\|(W_{r-1})_{\varphi,(\varphi+\varepsilon)}(f_n)\right\|_{\delta+1} \to 0$.

Theorm 3.4.Given $\varphi, (\varphi+\varepsilon)\epsilon\ H_0$ such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$ and $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is continuous on H_0, we have that the following assertions are equivalent:
i. The operator $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is compact on H_0.
ii. The operator $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is weakly compact on H_0.
iii. The measure $\mu_{\varphi,\varphi+\varepsilon,1}$satisfies.

$$\lim_{\tau\to 0} \sup_{b\in\mathbb{T}} \frac{\mu_{\varphi,\varphi+\varepsilon,1(S_{r-1}(b,\tau))}}{\tau} = 0$$

Proof.$(i) \Longrightarrow (ii)$. This is obvious.

$(ii) \Rightarrow (iii)$. We apply Proposition 3.2 with $X = Y = H_1, \tau_1$ the topology of the uniform convergence on compact sets, τ_1 the topology of the pointwise convergence, and, of course, $T_{r-1} = (W_{r-1})_{\varphi,\varphi+\varepsilon}$. It is clear that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is $\tau_1 - \tau_2$continuous. Consider the map $S_{r-1}: H_0 \to L_0(\overline{\mathbb{R}}, \mu_{\varphi,\varphi+\varepsilon,1})$given by $S_{r-1}(f) = f$. By Lemma 2.1, we have that $\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(h)\|_0 = \|S_{r-1}(h)\|_{L_{1(\overline{\mathbb{R}},\mu_{\varphi,\varphi+\varepsilon,1})}}$.for $h \epsilon H_0$. Since $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is weakly compact on H_0 , by Proposition 3.2, S_{r-1} is alsoweakly compact.

Now, suppose assertion (iii). is not satisfied. Then there are $\beta > 0, \tau_n \to 0 (0 < \tau_n < 1)$, and$b_n \in \mathbb{T}$ such that $\mu_{\varphi,\varphi+\varepsilon,1}(S_{r-1}(b_n, \tau_n)) \geq \beta r_n$. Let us denote $a_n = (1-\tau_n)b_n$and $f_n(z) = 1/(1-a_n z)^4$. Then $f_n \epsilon H_0$ and

$\|f_n\|_0 = \frac{1}{\tau_n^3}\frac{1+(1-\tau_n)^2}{(2+\tau_n)^3}$.

Now we take $g_n = f_n/\|f_n\|_0$. To get a contradiction, we are going to show that for each subsequence(g_{n_k}) , the sequence $S_{r-1}(g_{n_k})$is not weakly k kconvergent. By[14,15], it will be enough to get that the set $\{S_{r-1}(g_{n_k}): k\epsilon\mathbb{N}\}$is not uniformly integrable, i.e., there is $\varepsilon > 0$ such that for every $\eta > 0$ there exists a measurable subset A of $\overline{\mathbb{R}}$ and $k\epsilon\mathbb{N}$ such that $\eta_{\varphi,\varphi+\varepsilon,1}(A) \leq \eta$ and $\int_A |g_{n_k}| d\mu_{\varphi,\varphi+\varepsilon,1} \geq \varepsilon$. Take $\varepsilon = \beta/4$ and let us fix an arbitrary η. Since $\mu_{\varphi,\varphi+\varepsilon,1}$is a Carleson measure, there is a constant M such that $\mu_{\varphi,\varphi+\varepsilon,1}\left(S_{r-1}(b_n, r_{n_k})\right) \leq \eta$ for all $b\epsilon\mathbb{T}$ and $0 < \tau < 10$. So, we can take$k$ such that $\mu_{\varphi,\varphi+\varepsilon,1}\left(S_{r-1}(b_{n_k}, \tau_{n_k})\right) \leq \eta$. On the other hand, bearing in mind that $|f_{n_k}(z)| \geq (2\tau_{n_k})^{-4}$ whenever $z \in S_{r-1}(b_{n_k}, \tau_{n_k})$, we have that

$$\int_{S_{r-1}(b_{n_k},\tau_{n_k})} |g_{n_k}| d\mu_{\varphi,\varphi+\varepsilon,1} \geq \frac{(2\tau_{n_k})^{-4}}{\|f_{n_k}\|_1}\mu_{\varphi,\varphi+\varepsilon,1}\left(S_{r-1}(b_{n_k}, \tau_{n_k})\right) \geq \frac{(2\tau_{n_k})^{-4}}{\|f_{n_k}\|_1}\beta\tau_{n_k} \geq \frac{\beta}{4}.$$

$(iii) \Rightarrow (i)$. We will apply Lemma 3.3. Before doing this, we have to introduce an auxiliary Carleson measureμ. By (iii),

$$\lim_{\tau\to 0} \sup_{b\in\mathbb{T}} \frac{\mu_{\varphi,\varphi+\varepsilon,1}\left(S_{r-1}(b_{n_k}, \tau_{n_k})\right)}{\tau} = 0$$

Then we also have that

$$\lim_{\tau\to 0} \sup_{b\in\mathbb{T}} \frac{\mu_{\varphi,\varphi+\varepsilon,1}(W_{r-1}(b,\tau))}{\tau} = 0$$

where $(W_{r-1})(b,\tau)$. are the Carleson windows in $\overline{\mathbb{R}}$ given by

$$W_{r-1}(b,\tau) = \{\mathfrak{C}e^{i\theta} \epsilon\mathbb{R}: 1-\tau \leq \mathfrak{C} \leq 1, |\theta - t| \leq \tau\}$$

Where $= e^{it}$. Given $\varepsilon > 0$, we may find τ_0 such that $\mu_{\varphi,\varphi+\varepsilon,1}(W_{r-1}(b,\tau)) \leq 2\varepsilon\tau$ for all $b \in \mathbb{T}$ and $\tau \leq \tau_0$. Let us define the measure μ given by

$$\mu(E) \coloneqq \mu_{\varphi,\varphi+\varepsilon,1}(E \cap \{z \in \overline{\mathbb{R}}: 1-\tau_0 \leq |z|\} \leq 1)$$

Then μ is a Carleson measure on $\overline{\mathbb{R}}$ with $\mu(W_{r-1}(b,\tau)) \leq 2\varepsilon r$ for $0 < \tau < 1(see[3])$. So, by[3] , there is a constant C (independent of ε).

such that

$\int_{\overline{\mathbb{R}}} |f|\, d\mu \leq C\varepsilon\|f\|_1$,

for all $f \in H_1$.

Once we have built the measure μ , we are going to apply Lemma 3.3 to get that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is compact on H_0. Take (f_n). a sequence in H_0 such that $(f_n) \to 0$ uniformly on compact subsets of $\mathbb{R}$ and $\|f_n\|_1 \leq 1$. Then, by Lemma 2.1,

Since $(f_n) \to 0$ uniformly on compact subsets of $\mathbb{D}$, there is n_0 such that if $n \epsilon \mathbb{N}$ and $n \geq n_0$ we have that $|f_n(z)| \leq \varepsilon/\mu_{\varphi,\varphi+\varepsilon,1}\left((1-r_0)\overline{\mathbb{R}}\right)$. for all $z \epsilon (1-\tau_0)\overline{\mathbb{R}}$.

So

$$\int_{(1-\tau_0)\overline{\mathbb{R}}} |f_n| d\mu_{\varphi,\varphi+\varepsilon,0} \leq \frac{\varepsilon}{\mu_{\varphi,\varphi+\varepsilon,0((1-\tau_0)\overline{\mathbb{R}})}}\mu_{\varphi,\varphi+\varepsilon,1}\left((1-\tau_0)\overline{\mathbb{R}}\right) = \varepsilon$$

On the other hand, we have that

$$\int_{\overline{\mathbb{R}}/(1-\tau_0)\mathbb{R}} |f_n| d\mu_{\varphi,\varphi+\varepsilon,0} = \int_{\overline{\mathbb{R}}/(1-r_0)\mathbb{R}} |f_n| d\,\mu = \int_{\overline{\mathbb{R}}} |f_n| d\mu \le C\varepsilon \|f_n\|_0 \le C\varepsilon$$

Hence $\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon} f_n\right\|_1 \le (1+C)\varepsilon$.

Theorem 3.5.Fixing $\delta > 0$ and given $\varphi, \varphi+\varepsilon \in H_{\delta+1}$such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$ and $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is continuous on $H_{\delta+1}$, we have that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is compact on $H_{\delta+1}$ if and only if

$$\lim_{\tau\to 0} \sup_{b\in\mathbb{T}} \frac{\mu_{\varphi,\varphi+\varepsilon,\delta+1}\big(S_{r-1}(b,\tau)\big)}{\tau} = 0$$

Proof. Suppose that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is compact on $H_{\delta+1}$ and that there are $\beta > 0$, $\tau_0 \to 0 (0 < \tau_0 < 1)$, and $b_n \in \mathbb{T}$such that$\mu_{\varphi,\varphi+\varepsilon,\delta+1}\big(S_{r-1}(b,\tau)\big) \ge \beta\tau_n$. Let us enote $a_n = (1-\tau_n)b_n$and $f_n(z) = 1/(1-a_n z)^{4/\delta+1}$. Then $f_n \in H_{\delta+1}$ and

$$\|f_n\|_{\delta+1}^{\delta+1} = \frac{1}{\tau_n^3} \frac{1+(1-\tau_n)^2}{(2+\tau_n)^3}$$

Now we take $g_n = f_n / \|f_n\|_{\delta+1}$. By [3], $g_n$converges to zero uniformly on compact subsets of $\mathbb{R}$. By Lemma 3.3, to get that $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is not compact, we have just to show that $\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(g_n)\right\|_{\delta+1}$does not converge to zero. Arguingas in the proof of Theorem 3.4, we have that

$$\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(f_n)\right\|_{\delta+1}^{\delta+1} = \int_{\mathbb{T}} |\varphi+\varepsilon|^{\delta+1} |g_n \circ \varphi|^{\delta+1} dm = \int_{\overline{\mathbb{R}}} |g_n|^{\delta+1} d\mu_{\varphi,\varphi+\varepsilon,\delta+1}$$

$$\ge \int_{S(b_n,\tau_n)} |g_n|^{\delta+1} d\mu_{\varphi,\varphi+\varepsilon,\delta+1} \ge \frac{(2\tau_n)^{-4}}{\|f_n\|_{\delta+1}^{\delta+1}} d\mu_{\varphi,\varphi+\varepsilon,\delta+1}\big(S_{r-1}(b_n,\tau_n)\big) \ge \frac{(2\tau_n)^{-4}}{\|f_n\|_{\delta+1}^{\delta+1}} \beta\tau_n \ge \frac{\beta}{4}$$

The other implication can be obtained by following the same steps as in the proof of Theorem 3.4.

IV. COMPLETE CONTINUITY

In this part we characterize the case where $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is a completely continuous operators series. Its proof is a slight generalization of [1,15].

Theorem 4.1.Given $\varphi, \varphi+\varepsilon \in H_0$ such that $\varphi(\mathbb{R}) \subseteq \mathbb{R}$ and $(W_{r-1})_{\varphi,\varphi+\varepsilon}$ is continuous on H_0, we have $(W_{r-1})_{\varphi,\varphi+\varepsilon}$is completely continuous on H_0if and only if $\varphi+\varepsilon = 0$almost everywhere in $\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}$.

Proof. Let f be a function in $L_\infty(\mathbb{T}, m)$. By the Riemann-Lebesgue lemma, the sequence given by its Fourier coefficients is in c_0 , so we have

that $\int_{\mathbb{T}} f(z)\bar{z}^n \, dm \to 0$ as $n \to \infty$. Equivalently, the sequence (z^n)converges to 0 in the weak topology of $L_0(\mathbb{T}, m)$. and, hence, in H_0. Therefore, $\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(z^n)\right\|_0 \to 0$. Moreover,

$$\int_{\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}} |\varphi+\varepsilon| \, \mathrm{dm} = \int_{\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}} |\varphi+\varepsilon||\varphi|^n \mathrm{dm} = \left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(z^n)\right\|_0$$

Hence$\int_{\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}} |\varphi+\varepsilon|$ dm=0, and we get that $\varphi+\varepsilon = 0$ almost everywhere on the set$\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}$. Conversely, let (f_n). be a weakly null sequence in H_0. Since $(f_n(z)) \to 0$ for all $z \in \mathbb{R}$ and $\varphi+\varepsilon$=0 almost everywhere in $\{e^{i\theta} \epsilon \mathbb{T}: \varphi(e^{i\theta}) \epsilon \mathbb{T}\}$, we have that $(W_{r-1})_{\varphi,\varphi+\varepsilon}(f_n)$. goes to zero pointwise almost everywhere on the unitcircle .In particular, the sequence $(W_{r-1})_{\varphi,\varphi+\varepsilon}(f_n)$. converges in measure to zero in $L_0(\mathbb{T}, m)$. Moreover, $(W_{r-1})_{\varphi,\varphi+\varepsilon}(f_n)$ goes to zero in the weak topology of $H_0$and, so, in the weak topology of $L_0(\mathbb{T}, m)$. Finally, bearing in mind that a sequence in $L_0(\mathbb{T}, m)$converges to zero in the norm topology whenever itconverges to zero in measure and in the weak topology see [6,15], we have that$\left\|(W_{r-1})_{\varphi,\varphi+\varepsilon}(f_n)\right\|_0 \to 0$.

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*Yousif .A. Mohammed. “Bounded Composition Operators on Hardy Spaces.” International Journal Of Engineering Research And Development , vol. 13, no. 11, 2017, pp. 62–67.