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On the Zeros of A Polynomial Inside the Unit Disc

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ABSTRACT: In this paper we find the number of zeros of a polynomial inside the unit disc under certain conditions on the coefficients of the polynomial.
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I. INTRODUCTION

In the context of the Enestrom-Kakeya Theorem [4] which states that all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with $a_n \ge a_{n-1} \ge ......\ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$, Q. G. Mohammad [5] proved the following result giving a bound for the number of zeros of P(z) in $|z| \le \frac{1}{2}$:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ......\ge a_1 \ge a_0 > 0,$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed $1+\frac{1}{\log 2}\log\frac{a_n}{a_0}$.

Various bounds for the number of zeros of a polynomial with certain conditions on the coefficients were afterwards given by researchers in the field (e.g. see [1],[2],[3]).

II. MAIN RESULTS

In this paper we find a bound for the number of zeros of a polynomial in a closed disc of radius less than 1 and prove

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j, \operatorname{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_{\lambda+1} \ge \tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then the number of zeros of P(z) in $|z| \le \delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+(k-1)|\alpha_n|+k\alpha_n-\tau\alpha_\lambda+L+(1-\tau)|\alpha_\lambda|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Taking a_j real i.e. $\beta_j = 0, \forall j=0,1,2,.....,n$, Theorem 1 reduces to the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$ka_n \ge a_{n-1} \ge ......\ge a_{\lambda+1} \ge \tau a_\lambda$$

and

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|$$

Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|a_n|+a_n)-\tau a_\lambda+L+(1-\tau)|a_\lambda|}{|a_0|}.$$

Taking $\tau=1$ in Cor. 1 , we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k\ge 1$,

$$ka_n\ge a_{n-1}\ge......\ge a_{\lambda+1}\ge a_\lambda$$

and

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|$$

Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|a_n|+a_n)-a_\lambda+L}{|a_0|}.$$

Taking $k=1$ in Cor. 1 , we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$ and for some $o<\tau\le 1$,

$$a_n\ge a_{n-1}\ge......\ge a_{\lambda+1}\ge \tau a_\lambda$$

and

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|$$

Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+a_n-\tau a_\lambda+L+(1-\tau)|a_\lambda|}{|a_0|}.$$

Taking $\tau=1$ in Theorem 1 , we get the following result:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k\ge 1$, ,

$$k\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_{\lambda+1}\ge\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+(k-1)|\alpha_n|+k\alpha_n-\alpha_\lambda+L+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Taking $k=1$ in Theorem 1 , we get the following result:

Corollary 5: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0\le\lambda\le n-1$ and for some $o<\tau\le 1$,

$$\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_{\lambda+1} \ge \tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|.$$

Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+\alpha_n-\alpha_\lambda+L+(1-\tau)|\alpha_\lambda|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Similarly for other different values of the parameters, we get many other interesting results.

III. LEMMA

For the proof of Theorem 1, we need the following result:

Lemma: Let f (z) be analytic for $|z|\le 1, f(0)\ne 0$ and $|f(z)|\le M$ for $|z|\le 1$. Then the number of zeros of f(z) in $|z|\le\delta, 0<\delta<1$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$.

(for reference see [6]).

IV. PROOF OF THEOREM 1

Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\
&=(1-z)(a_n z^n+a_{n-1}z^{n-1}+......+a_1 z+a_0)\\
&=-a_n z^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda\\
&\quad+......+(a_1-a_0)z+a_0\\
&=-a_n z^{n+1}-(k-1)\alpha_n z^n+(k\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......+(\alpha_{\lambda+1}-\tau\alpha_\lambda)z^{\lambda+1}\\
&\quad+(\tau-1)\alpha_\lambda z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+......+(\alpha_1-\alpha_0)z+\alpha_0+i\{(\beta_n-\beta_{n-1})z^n\\
&\quad+......+(\beta_1-\beta_0)z+\beta_0\}\end{aligned}$$

For $|z|\le 1$, we have, by using the hypothesis

$$\begin{aligned}|F(z)|&\le|a_n|+(k-1)|\alpha_n|+|k\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+......+|\alpha_{\lambda+1}-\tau\alpha_\lambda|+(1-\tau)|\alpha_\lambda|\\
&\quad+|\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-\alpha_0|+|\alpha_0|+|\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|.+.......\\
&\quad+|\beta_1-\beta_0|+|\beta_0|\\
&\le|a_n|+(k-1)|\alpha_n|+k\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{\lambda+1}-\tau\alpha_\lambda+(1-\tau)|\alpha_\lambda|\\
&\quad+|\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-\alpha_0|+|\alpha_0|+|\beta_n|+|\beta_{n-1}|+|\beta_{n-1}|.+|\beta_{n-2}|+.......\\
&\quad+|\beta_1|+|\beta_0|+|\beta_0|\\
&\le|a_n|+(k-1)|\alpha_n|+k\alpha_n+|\alpha_\lambda|+L-\tau(|\alpha_\lambda|+\alpha_\lambda)+2\sum_{j=0}^{n}|\beta_j|\end{aligned}$$

Since F(z) is analytic for $|z|\le 1$, $F(0)=a_0\ne 0$, it follows by the Lemma that the number of zeros of F(z) in $|z| \le \delta, 0 < \delta < 1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{|a_n| + (k-1)|\alpha_n| + k\alpha_n + |\alpha_\lambda| + L - \tau(|\alpha_\lambda| + \alpha_\lambda) + 2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Since the zeros of P(z) are also the zeros of F(z) , it follows that the number of zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{|a_n| + (k-1)|\alpha_n| + k\alpha_n + |\alpha_\lambda| + L - \tau(|\alpha_\lambda| + \alpha_\lambda) + 2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

That completes the proof of Theorem 1.

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