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A new approach for solving fractional partial Differential equations via a collocation method based on Muntz-Legendre polynomials

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Abstract:
In this paper, a numerical method for calculating the approximate solution of fractional partial differential equations (FPDEs) using a collocation method based on Muntez-Legendre polynomials has been presented. Having specific properties, Muntz-Legendre polynomials are one of the most suitable basis for solving partial differential equations with fractional orders via a collocation method. In this method, we convert FPDEs to a system of algebraic equations, which can be solved by computer using symbolic methods. The presented method is more accurate and more efficient than other methods presented in other papers, we examine this issue using a number of examples.
Keywords: *Fractional partial differential equations, collocation method, Muntz-Legendre polynomials*

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I. INTRODUCTION:

At the outset of the new millennium, fractional calculus is known as one of the most important parts of computational science and engineering science. Due to the close correlation between fractional calculus and fractal geometry, fractional computation helps us to have better knowledge of physical and natural phenomena, which are well mentioned in [1]. A lot of research has been done in recent years, and there are many articles on the importance and applications of fractional calculus. For example, some of these articles can be found in [2-10].

It should be noted that fractional partial differential equations (FPDE) have attracted many researchers in recent years in mathematics, physics, chemistry and applied sciences [11-17]. It is very difficult to obtain an analytical solution for these equations and that is why mathematicians have provided various numerical methods for obtaining approximate solutions of FPDEs. For example, Umer Saeed, Mujeeb ur Rehman [18] used operational matrices of Har wavelets and Picard's technique has been used to obtain the approximate solution of the nonlinear fractional partial differential equations. Lifeng Wang et al. [19], using the Har wavelet series and an operational matrix, transformed FPDEs into Sylvester equations, and presented a numerical method for solving these equations. In [20,21], authors used the Tau method to solve FPDEs. In [22] Abbas Saadatmandi et al. Used the Sinc-legendre collocation method to solve these equations. In [23], Second Kind Shifted Chebyshev polynomials are used to obtain an approximate solution for such equations. In [24], Q.Liu et al. presented a mesh less-based method based on interpolation to solve these equations. [25] Heydari et al. have provided a computational method, using the Legendre wavelets, for obtaining the solution of partial differential equations with a boundary condition, and some other methods used to solve these equations can be found in [26-40].

One common method for solving differential equations is the use of orthogonal polynomials such as Legendre polynomials, Chebyshev, etc. [28,29]. Esmaeili et al. [41], introduced Muntez Legendre's orthogonal polynomials. An important feature that distinguishes these polynomials from classical orthogonal polynomials is that these polynomials have fractional powers. This makes the use of these functions very suitable for solving fractional differential equations. In this paper, we use Muntez Legendre polynomials to solve FPDEs as follows,

$$\begin{cases} a(x)\dfrac{\partial^{\alpha} u(x,t)}{\partial^{\alpha} x} + \dfrac{\partial^{\beta} u(x,t)}{\partial^{\beta} t} + b(x)\dfrac{\partial^{2} u(x,t)}{\partial^{2} x} = f(x,t) \\ u(x,0) = g_0(x), u(0,t) = g_1(t) \\ 0 \le x \le T, 0 \le t \le L \end{cases} \qquad (1)$$

Where $a(x), b(x), f(x,t), g_0(x), g_1(t)$ are continous functions and $0 < \alpha < 2, 0 < \beta \le 1$.

Presented method in this paper is a collocation method that can be easily implemented using computational computer programs. A summary of the work done in this paper is as follows. In the second section, we summarize the fractional calculations and provide the definitions needed to calculate the fractional derivative. In the third section, we define the Jacobi and Muntz-Legendre polynomials and provide a recursive relation for obtaining theMuntz- Legendre polynomials, and also we examine some of the features of these polynomials. In the fourth section, we present a numerical method for obtaining the solution of the partial differential equation (FPDE). In the fifth section, using a number of examples, we evaluate the method described in Section 4 and compare the results obtained using this method with the results obtained in other articles.

II. FRACTIONAL CALCULUS

Fractional calculations are one of the very old issues in computational science. In fact, for the first time in 1695, by Leibniz, fractional calculus has been cited, since then so much work has been done in this field. A history of fractional calculations can be found in [42].

In this paper, we use the Caputo formula to compute the fractional derivative of a function that is defined as follow:

$$D_*^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \dfrac{f^{(m)}(x)}{(x-\tau)^{\alpha-m+1}} d\tau & m-1 \le \alpha < m \\ \dfrac{d^m f(x)}{dx^m} & \alpha = m \end{cases} \qquad (2)$$

Where:

$\alpha > 0, x > 0, m \in \mathbb{N}$

Some features of the Caputo fractional operator are as follows:

i) For every constant number C , $D_*^{\alpha} C = 0$

ii) For every function such as $f(x) = x^{\mu}$,we have,

$$D_*^{\alpha} f(x) = \begin{cases} 0 & \mu \in \mathbb{N}_0, \mu < \lceil \alpha \rceil \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} x^{\mu-\alpha} & \mu \in \mathbb{N}_0, \mu \ge \lceil \alpha \rceil \, or \, \mu \notin \mathbb{N}_0, \mu > \lceil \alpha \rceil \end{cases}$$

Where

$\mathbb{N}_0 = \{0,1,2,\ldots\}$

$\lceil \alpha \rceil$ is smallest integer number which is bigger than alpha

iii) Caputo fractional derivative operator is a linear operator, it means that for every constants $\{c_i\}_{i=0}^{n}$ we have:

$$D_*^{\alpha} \left(\sum_{i=0}^{n} c_i f_i(x) \right) = \sum_{i=0}^{n} c_i D_*^{\alpha} f_i(x)$$

2.1 Jacobi polynomials

These polynomials are orthogonal with respect to the weight function $w_{(x)}^{(\alpha,\beta)} = (1-x)^{\alpha}(1+x)^{\beta}$ in the interval $[-1,1]$, where $\alpha, \beta > -1$, and can be obtained by the following recursive formula

$$J_0^{(\alpha,\beta)}(x)=1 \qquad J_1^{(\alpha,\beta)}(x)=\frac{1}{2}((\alpha-\beta)+(\alpha+\beta+2)x)$$
$$a_{1,k}^{(\alpha,\beta)}J_{k+1}^{(\alpha,\beta)}(x)=a_{2,k}^{(\alpha,\beta)}(x)J_{k}^{(\alpha,\beta)}(x)-a_{3,k}^{(\alpha,\beta)}J_{k-1}^{(\alpha,\beta)}(x)$$
$$a_{1,k}^{(\alpha,\beta)}=2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)$$
$$a_{2,k}^{(\alpha,\beta)}(x)=(2k+\alpha+\beta+1)((2k+\alpha+\beta)(2k+\alpha+\beta+2)x+\alpha^2-\beta^2) \qquad (3)$$
$$a_{3,k}^{(\alpha,\beta)}=2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)$$

2.2 Muntz- Legendre polynomials

Let $\Lambda_n=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ be a set with $\mathrm{Re}(\lambda_k)>-\frac{1}{2}$, then Muntz-Legendre polynomials can be defined on the interval (0,1] as follow;

$$P_n(x)=P_n(x:\Lambda_n)=\sum_{k-0}^{n}C_{n,k}x^{\lambda_k}, \qquad C_{n,k}=\frac{\prod_{v=0}^{n-1}(\lambda_k+\overline{\lambda_v}+1)}{\prod_{v=0,v\neq k}^{n}(\lambda_k-\lambda_v)} \qquad (4)$$

Here we set positive number alpha such that
$\lambda_k=\alpha k$

In this way, the Muntz-Legendre polynomials on the interval [0,T] can be defined as follow,

$$L_n(x:\alpha):=\sum_{k=0}^{n}C_{n,k}\left(\frac{x}{T}\right)^{\alpha k}, \qquad C_{n.k}=\frac{(-1)^{n-k}}{\alpha^n k!(n-k)!}\prod_{v=0}^{n-1}((k+v)\alpha+1) \qquad (5)$$

For example let
$\alpha=1/4, \quad T=1$
Then
$L_{10}(x:\alpha)$
Is as bellow,

$$-10010.00000x^{1/4}+1.351350000\,10^5\sqrt{x}-9.609599998\,10^5x^{3/4}-1.102701600\,10^7x^{5/4}$$
$$+1.939938000\,10^7x^{3/2}-2.217072000\,10^7x^{7/4}+4.084079998\,10^6x$$
$$+1.587222000\,10^7x^2+1.144066000\,10^6x^{5/2}-6.466459994\,10^6x^{9/4}+286.0000000$$

As you can see, and it is mentioned in [41,43], the coefficients of these polynomials are greatly enlarged with enlargement n, so in order to avoid the computational error Esmaeili et al. [41] used a recursive relation based on Jacobi polynomials to obtain Muntz-Legendre polynomials which is a stable recursive formula for obtaining these polynomials as follows:

$$L_0(x:\alpha)=1, \qquad L_1(x:\alpha)=\left(\frac{1}{\alpha}+1\right)\left(\frac{x}{T}\right)^{\alpha}-\frac{1}{\alpha}$$

$$b_{1,n}L_{n+1}(x:\alpha)=b_{2,n}(x)L_n(x:\alpha)-b_{3,n}L_{n-1}(x:\alpha), \qquad (6)$$

$$b_{1,n}=a_{1,n}^{(0,\frac{1}{\alpha}-1)}, \qquad b_{2,n}(t)=a_{2,n}^{(0,\frac{1}{\alpha}-1)}\left(2\left(\frac{x}{T}\right)^{\alpha}-1\right), \qquad b_{3,n}=a_{3,n}^{(0,\frac{1}{\alpha}-1)}$$

Also we have the following relation for these polynomials,

$$\int_0^T L_n(x:\alpha)L_m(x:\alpha)dx=\frac{T\delta_{m,n}}{1+2\alpha n}, \qquad (7)$$

2.3. Approximation of two-variable functions using the Muntz-Legendre polynomials

let $u(x,t)$ be a known two variable function and approximation of this function is required in the span $[0,T]\times[0,L]$. At first, for this aim, let $\{L_i(x:\alpha)\}_{i=0}^{n}$ has been defined on the interval [0,T] , and $\{L_j(t:\beta)\}_{j=0}^{m}$ Has been defined on the [0,L] and $\alpha,\beta>0$. Then the approximation function can be defined as bellow:

$$u(x,t)\approx\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}L_i(x,\alpha)L_j(t,\beta) \qquad (8)$$

Now based on (7) Unknown coefficients $C_{i,j}$ can be obtained as follow

$$C_{i,j}=\frac{2i\alpha+1}{T}\times\frac{2j\beta+1}{L}\int_0^T\int_0^L u(x,t)L_i(x:\alpha)L_j(t:\beta)dtdx \qquad i=0..n \qquad j=0..m$$

III. NUMERICAL SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

In this section we present a numerical method based on the Muntez-Legendre polynomials to obtain the approximate solution of equation (1) as follows.

$$u(x,t)\approx\tilde{u}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}L_i(x,\alpha)L_j(t,\beta) \qquad (9)$$

By substituting (9) with (1) we have:

$$\begin{cases} a(x)\dfrac{\partial^{\alpha}\tilde{u}(x,t)}{\partial^{\alpha}x}+\dfrac{\partial^{\beta}\tilde{u}(x,t)}{\partial^{\beta}t}+b(x)\dfrac{\partial^{2}\tilde{u}(x,t)}{\partial^{2}x}=f(x,t) \\ \tilde{u}(x,0)=g_0(x),\tilde{u}(0,t)=g_1(t) \\ 0\le x\le T,0\le t\le L \end{cases} \qquad (10)$$

Note that $\frac{\partial^{\alpha}\tilde{u}(x,t)}{\partial^{\alpha}x},\frac{\partial^{\beta}\tilde{u}(x,t)}{\partial^{\beta}t}$ are Caputo fractional derivative operators for u(x,t) with respect to x and t , respectively. Furthermore, it is notable that in (10) $C_{i,j}$ are Unknown coefficients, that should be found. For this reason, nodes $(\theta_p,\eta_q)\in[0,T]\times[0,L]$ $P=0,1,..n$ $q=0,1,..m$, which can be obtained as follows

$$\theta_0 = \frac{T}{n+1}, \quad \theta_p = \theta_{p-1} + \theta_0 \qquad p = 1, 2, \ldots, n$$

$$\eta_0 = \frac{L}{m+1}, \quad \eta_q = \eta_{q-1} + \eta_0 \qquad q = 1, 2, \ldots, m$$

Thus, equation (10) is converted as follows:

$$\begin{cases} a(\theta_p) \dfrac{\partial^\alpha \tilde{u}(x,t)}{\partial^\alpha x}\Big|_{(\theta_p,\eta_q)} + \dfrac{\partial^\beta \tilde{u}(x,t)}{\partial^\beta t}\Big|_{(\theta_p,\eta_q)} + b(\theta_p) \dfrac{\partial^2 \tilde{u}(x,t)}{\partial^2 x}\Big|_{(\theta_p,\eta_q)} = f(\theta_p, \eta_q) \\ \tilde{u}(\theta_p, 0) = g_0(\theta_p), \tilde{u}(0, \eta_q) = g_1(\eta_q) \end{cases} \tag{11}$$

So (11) is an algebraic equation system with (m+1)(n+1) equations and (m+1)(n+1) unknowns that can be obtained by a known method using a computer program and by substituting obtained coefficients in to (9) , an approximated solution for unknown function u(x,t) can be obtained.

IV. NUMERICAL EXAMPLES:

In this section, we will examine the method presented in Section 4 using a number of examples and results are illustrated. All calculations performed in this section are performed using Maple software and the accuracy of 40 digits is used.

Example 1: For the first example, consider the following equation [22,34]

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} + x \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2t^\beta + 2x^2 + 2$$

$$u(0,t) = 2\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} t^{2\beta} \qquad u(x,0) = x^2 \qquad 0 \le x \le 1, 0 \le t \le 1, 0 < \beta < 1$$

The exact answer for above equation is

$$u(x,t) = x^2 + 2\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} t^{2\beta}$$

In Table 1, the absolute value of the error obtained by the presented method in this paper for $\beta = 0.5, t = 0.5$ and the papers [25] and [22] are compared. In Fig. 1, the diagram of absolute error for $\beta = 0.75, m = n = 3$ is shown. Fig. 2 shows the absolute value error for different values of beta and $x = 0.4$

Table 1: Comparison of absolute error for $\beta = 0.5, t = 0.5$ (example 1)

x	[34]wavelt method m=64	[22]sinc-legendere m=25	Present method m=n=3
0.1	1.210×10^{-3}	6.462×10^{-6}	2.1×10^{-39}
0.2	1.259×10^{-3}	1.578×10^{-5}	3.1×10^{-39}
0.3	1.865×10^{-3}	2.272×10^{-5}	4.7×10^{-39}
0.4	7.412×10^{-3}	2.674×10^{-5}	7.0×10^{-39}
0.5	1.000×10^{-6}	2.759×10^{-5}	7.0×10^{-39}
0.6	7.460×10^{-3}	2.534×10^{-5}	1.0×10^{-38}
0.7	1.724×10^{-3}	2.035×10^{-5}	1.3×10^{-38}

0.8	4.990×10^{-3}	1.320×10^{-5}	8.0×10^{-39}
0.9	1.678×10^{-2}	4.653×10^{-6}	1.5×10^{-38}

Figure 1 Absolute Value for B=0.75 (Example 1)

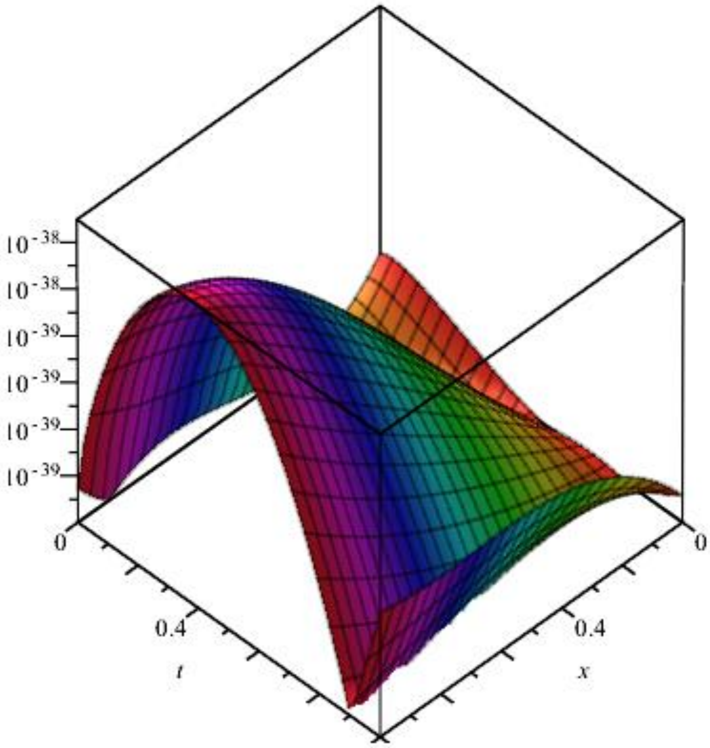
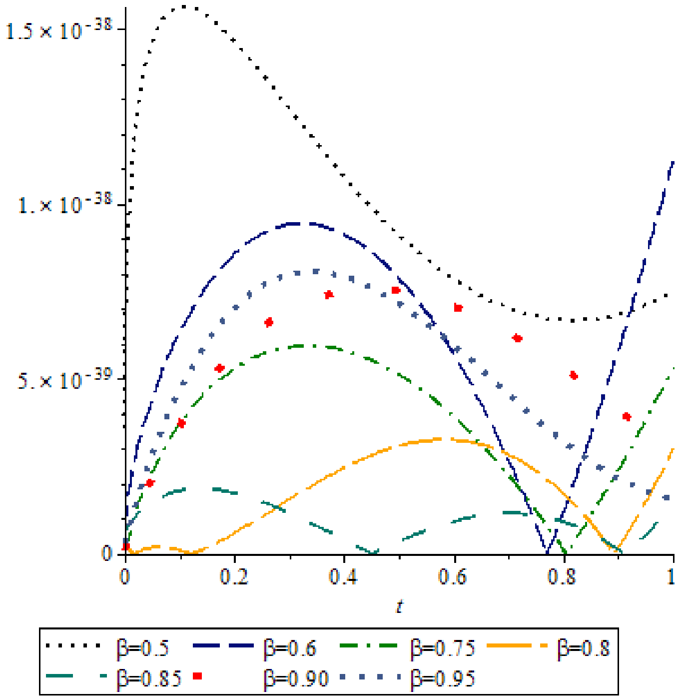


Figure 2: Absolute Error Chart for x=0.4 and Different B (Example 1)



Example2: Consider the following fractional partial differential equation [19]

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = f(x,t)$$

$$f(x,t) = 2\frac{\Gamma(3)\, x^{2-\alpha}(t^2+1)}{\Gamma(3-\alpha)} + 2\frac{\Gamma(3) t^{2-\beta}(x^2+1)}{\Gamma(3-\beta)}$$

$$u(x,0) = x^2+1 \quad u(0,t) = t^2+1$$

$$0 \le x \le 1, 0 \le t \le 1,$$

The exact solution for is

$$u(x,t) = (x^2+1)(t^2+1)$$

Table 2 compares the absolute error for Har wavelet method [19] and presented method for $\alpha = 1/2, \beta = 1/3$ and different values of x and t , also absolute errors for $\alpha = 1/2, \beta = 1/2$ and $\alpha = 1/3, \beta = 1/3$ are shown in table 2. As it can be seen from table 2, presented method provided more accurate results by less computational cost compared to Har wavelet method [19].

Table2: absolute errors fordifferent values of α, β for example 2

(x,t)	$\alpha = 1/2, \beta = 1/3$		$\alpha = 1/2, \beta = 1/2$	$\alpha = 1/3, \beta = 1/3$
	Haar wavelet [19]m=64	Present method m=n=6	Present method m=n=6	Present method m=n=6
$(0,0)$	1.366948e_009	6.80×10^{-34}	7.03×10^{-36}	7.38×10^{-34}
$(1/8,1/8)$	2.210144e_008	3.08×10^{-22}	5.73×10^{-22}	1.48×10^{-28}
$(2/8,2/8)$	3.298079e_008	1.23×10^{-21}	2.37×10^{-21}	6.15×10^{-28}
$(3/8,3/8)$	5.506236e_008	2.88×10^{-21}	5.61×10^{-21}	1.47×10^{-27}
$(4/8,4/8)$	8.024396e_008	5.44×10^{-21}	1.07×10^{-20}	2.81×10^{-27}
$(5/8,5/8)$	1.074419e_007	9.18×10^{-21}	1.81×10^{-20}	4.80×10^{-27}
$(6/8,6/8)$	1.363378e_007	1.45×10^{-20}	2.86×10^{-20}	7.64×10^{-27}
$(7/8,7/8)$	1.667197e_007	2.17×10^{-20}	4.30×10^{-20}	1.15×10^{-26}

Example3: Consider the following equation which is called (fractional heat- like equation) [45,46,22,44]

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{1}{2}x^2\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$u(0,t) = 0, u(x,0) = x^2$$

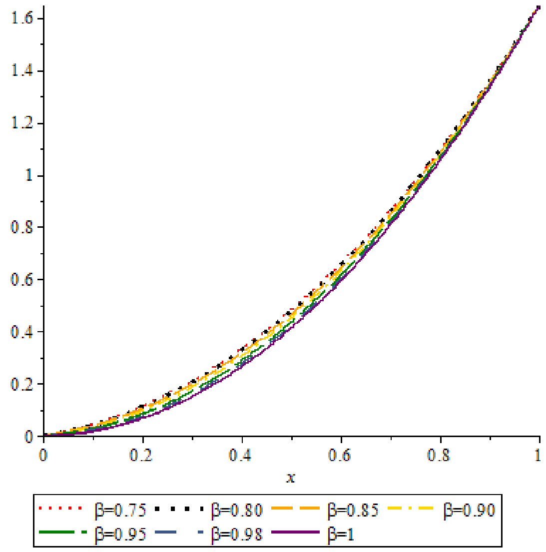
The exact solution for $\beta = 1$ is $u(x,t) = x^2 e^t$. Table 3 compares the absolute errors obtained by presented method and method in [22] . Figure 3 illustrates approximate solution obtained by presented method for $t = 0.5$ and different values of B . As it is obvious from Figure 3 , when B moves toward 1 the approximated solution converges toward the exact solution for B=1.

Table3: comparison of absolute errors obtained by presented method and [22]

t	x	[22]m=15	[22]m=25	m=n=7 present method	m=n=15 present method
0.25	0.3	1.09×10^{-6}	9.98×10^{-8}	3.26×10^{-9}	2.58×10^{-21}
	0.6	2.96×10^{-5}	2.70×10^{-6}	1.30×10^{-8}	1.03×10^{-20}

	0.9	9.94×10^{-5}	1.02×10^{-5}	2.93×10^{-8}	2.33×10^{-20}
0.5	0.3	6.45×10^{-6}	5.56×10^{-7}	4.20×10^{-9}	3.32×10^{-21}
	0.6	5.24×10^{-5}	4.87×10^{-6}	1.68×10^{-8}	1.33×10^{-20}
	0.9	1.47×10^{-4}	1.30×10^{-5}	3.78×10^{-8}	2.99×10^{-20}
0.75	0.3	1.40×10^{-5}	1.14×10^{-6}	5.39×10^{-9}	4.26×10^{-21}
	0.6	7.67×10^{-5}	6.90×10^{-6}	2.15×10^{-8}	1.70×10^{-20}
	0.9	2.01×10^{-4}	1.59×10^{-5}	4.85×10^{-8}	3.84×10^{-20}
1	0.3	1.83×10^{-5}	9.83×10^{-7}	4.25×10^{-9}	3.41×10^{-21}
	0.6	9.40×10^{-5}	8.40×10^{-6}	1.70×10^{-8}	1.36×10^{-20}
	0.9	4.26×10^{-4}	2.58×10^{-5}	3.82×10^{-8}	3.07×10^{-20}

Figure3: Approximated solutions for t-0.5 and different values of Β (example3)



Example4: Consider the following FPDE [23,21,37]

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + f(x,t)$$

$$u(x,0) = x^2(1-x), u(0,t) = 0, a(x) = \Gamma(1.2)x^{1.8}, \ f(x,t) = 3x^2(2x-1)e^{-t}$$

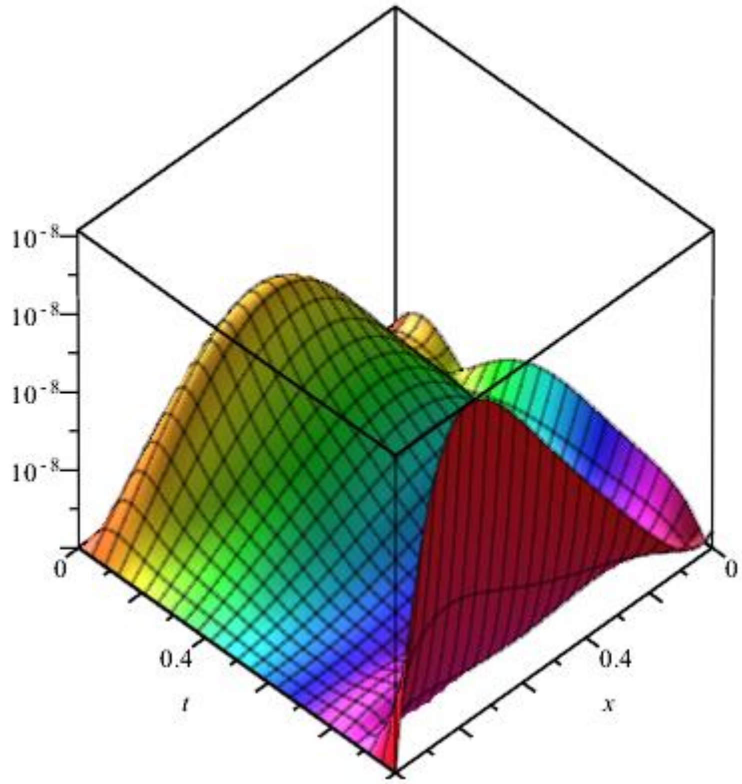
The exact solution for the above equation isa $u(x,t) = x^2(1-x)e^{-t}$ .Table 4 compares the absolute errors for t=1 and different x ,obtained by presented method ,with other methods in [21,23,37], also Figure 4 illustrates absolute error for $0 \le x, t \le 1$.

Table 4: absolute errors for t=1 and different x (example4)

x	[37]	[21]	[23]	Present method
0.1	4.26×10^{-5}	4.66×10^{-5}	5.46×10^{-6}	2.66×10^{-9}
0.2	5.39×10^{-5}	7.74×10^{-5}	8.51×10^{-6}	7.65×10^{-9}
0.3	6.12×10^{-5}	5.00×10^{-5}	9.60×10^{-6}	1.38×10^{-8}

0.4	6.48×10^{-5}	2.30×10^{-5}	9.18×10^{-6}	2.01×10^{-8}
0.5	6.45×10^{-5}	2.74×10^{-5}	7.69×10^{-6}	2.67×10^{-8}
0.6	5.98×10^{-5}	4.38×10^{-5}	5.60×10^{-6}	3.34×10^{-8}
0.7	5.23×10^{-5}	3.87×10^{-5}	3.33×10^{-6}	3.90×10^{-8}
0.8	4.48×10^{-5}	1.01×10^{-5}	1.34×10^{-6}	4.03×10^{-8}
0.9	3.91×10^{-5}	3.55×10^{-5}	8.39×10^{-8}	3.07×10^{-8}
1	2.81×10^{-5}	0	0	0

Figure 4: Absolute error for n=m=6 (example4)



Example5: Consider the following FPDE ,[23,45].

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + f(x,t)$$

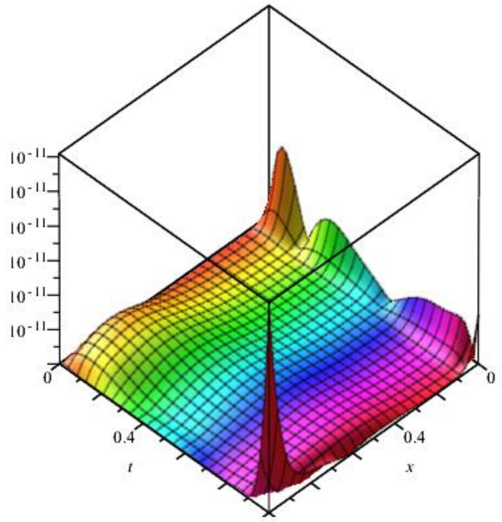
$$u(x,0) = x^3, u(0,t) = 0, a(x) = \frac{\Gamma(2.2)}{6}x^{2.8},\ f(x,t) = -(1+x)x^3 e^{-t}$$

The exact solution for this equation is $u(x,t) = x^2(1-x)e^{-t}$. Table 5 compares the obtained maximum absolute error by presented method and [23,47] for m=n=9 , t=1 . Furtheremore, Figure 5 shows the absolute error function for m=n=9.

Table 5: maximum absolute error for m=n=9 and t=1 (example 5)

Max error-extCN[47]	Max error [23]	Present method
6.84895×10^{-4}	8.3830×10^{-10}	6.0906×10^{-11}

Figure 5: Absolute error function for m=n=9 (Example 5)



Example 6: Consider the time-fractional Navier-Stokes equation as follow [48,49,50]

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = p + \frac{\partial^{2} u(x,t)}{\partial x^{2}} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} \qquad p \in R, 0 < \beta \leq 1$$

$$u(x,0) = 1 - x^{2}, u(1,t) = (p-4)\frac{t^{\beta}}{\Gamma(1+\beta)}$$

The exact solution for this equation is $u(x,t) = 1 - x^{2} + (p-4)\frac{t^{\beta}}{\Gamma(1+\beta)}$

Figure 6 illustrates the absolute error obtained by the Muntz-Legendre collocation method for P=1.5, B=0.75 and m=n=3. Also Figure 7 shows the approximated solution obtained by presented method for m=n=3 , p=1 ,x=1 and different values for B ,also Figure 8 illustrates absolute errors for obtained solutions which are illustrated in figure 7 .

Figure 6: Absolute error function for P=1.5, B=0.75, m=n=3 (Example 6)

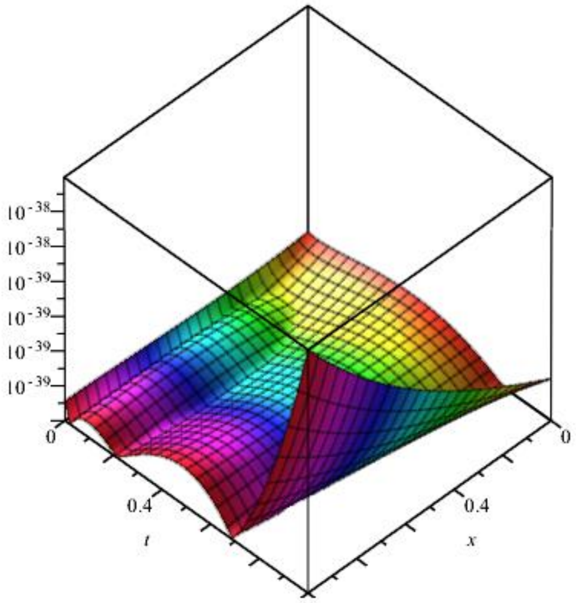


Figure 7: Approximated solutions for m=n=3, p=1, x=1 and different values of B

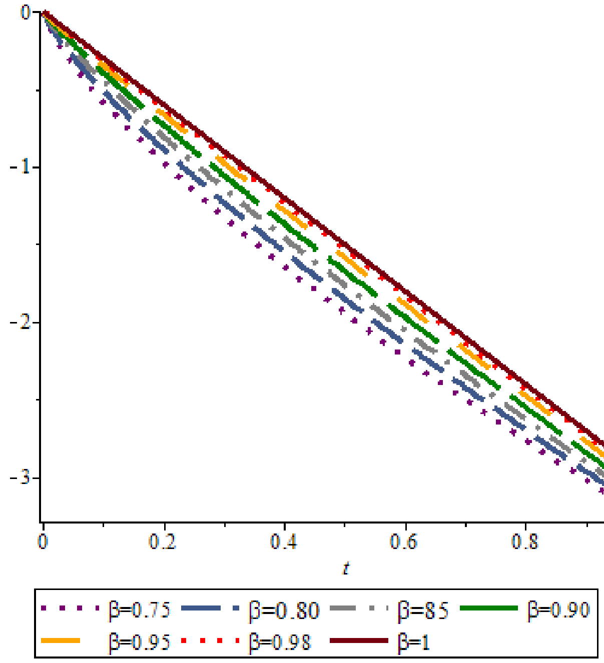
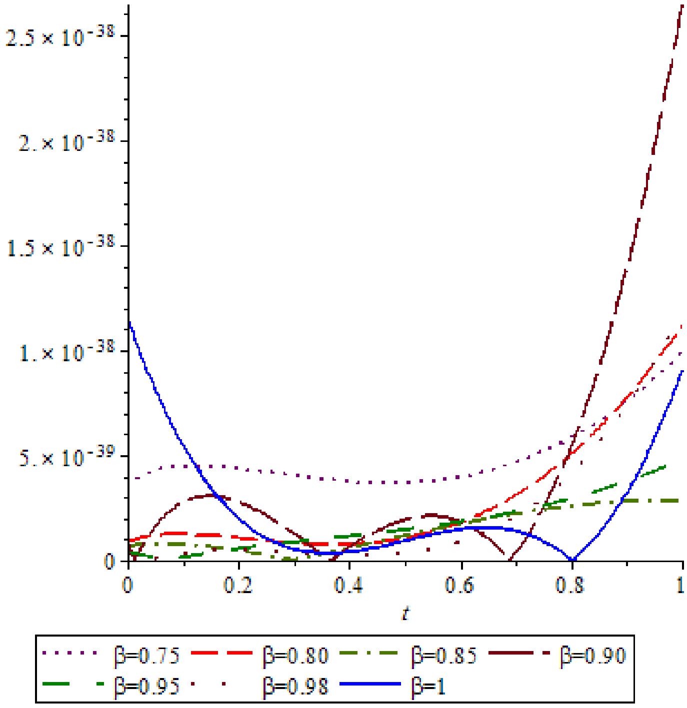


Figure 8: Absolute errors for approximated solutions illustrated in figure 7.



Example 7: For the last example consider the following FPDE

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} + \frac{\partial u(x,t)}{\partial x} = f(x,t)$$

$f(x,t) = \sin(x+t)$

$u(x,L) = \sin(x)\sin(T), u(T,t) = \sin(T)\sin(t)$

$x, t \in [0,T] \times [0,L]$

$0 < \beta \leq 1$

The exact solution for B=1 IS

$u(x,t) = \sin(x)\sin(t)$

Table 6 shows Maximum absolute errors for B=1 and different values of m, n. It can be seen from table 6 that as m, n increased, more accurate approximated solutions are obtained. Figure 9 shows the obtained approximated solutions by presented method for different values of B and

$m = n = 10, L = T = 1, t = 0.5$

As it is obvious from figure 10, when $\beta \to 1$ approximated solutions converge to the exact solution for B=1 .

Table6: Maximum absolute error for different values m, n and B=1 (example7)

$m = n =$	3	5	7	10	13	16
Max error	1.4×10^{-3}	6.4×10^{-6}	1.6×10^{-8}	1.2×10^{-12}	2.0×10^{-17}	4.5×10^{-22}

Figure 9: Approximated solution for m=n=10, L=T=2pi and B=1 (Example 7)

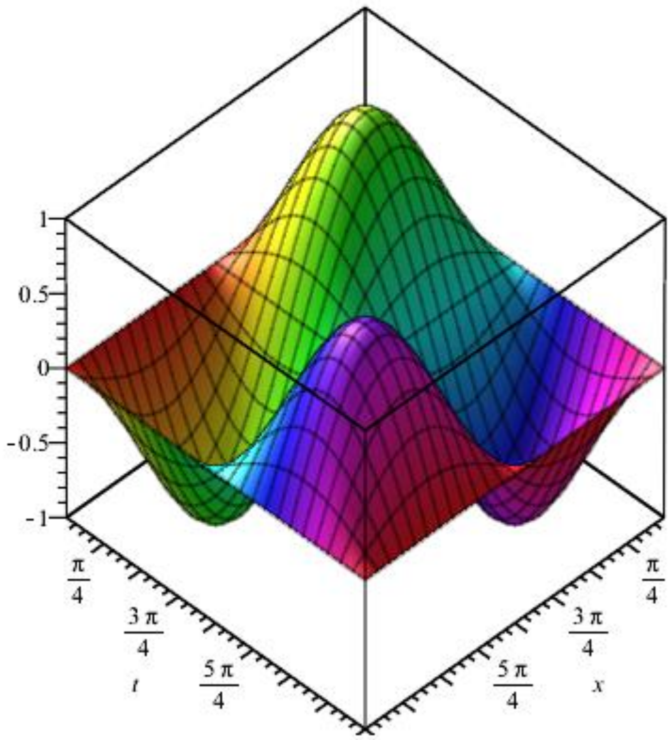
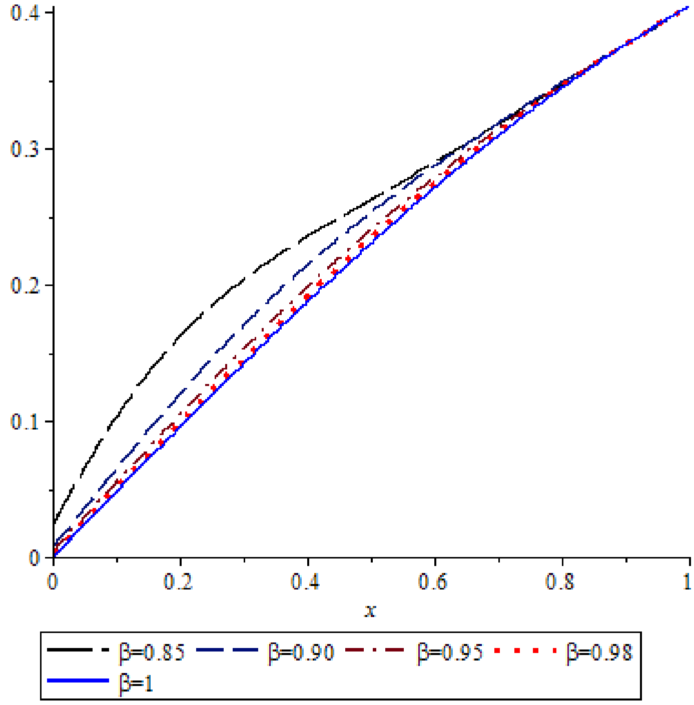


Figure 10: Approximated solutions for t=0.5 and different values of B (Example7)



V. CONCLUSION:

In this paper, a collocation method based on Muntz-Legendre polynomials has been used to solve partial differential equations with fractional derivatives, also Caputo formula to calculate fractional derivatives is considered. Furthermore, various examples are solved by the presented method and results are illustrated. As the results show for numerical examples, the presented method in this paper has a high degree of accuracy and efficiency compared to other methods. By using this method and fewer calculations, more accurate answers can be obtained than other methods.

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