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Solution of Riccati Equation using Lie Symmetry Method

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ABSTRACT: *Riccati equation is a non-linear ordinary differential equation of first order. It is used in different areas in Mathematics like theory of conformal mapping, algebraic geometry and also in Physics. This equation can be solved by converting it first into Bernoulli equation and then into linear differential equation. Lie symmetry method is one of those methods which are used to find general solution of differential equations of any order by finding lie symmetries as solutions of Lie Invariance condition. In this paper, this method is used to solve Riccati equation. The method is explained with the help of examples.*
KEYWORDS: *Lie Symmetry Method, Riccati Equation, Invariance Condition, Similarity Solutions*

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I.INTRODUCTION

Riccati equation is a non-linear ordinary differential equation of first order.Mathematicians and Scientists have devised different methods of solving Riccati equations.Sophus Lie, a Norwegian Mathematician, worked on transforming a given differential equation to such form which is easily integrable. He called these transformations as symmetries as the given differential equation remains invariant under such transformations. These symmetries not only transform the variables in the equation but also transforms their derivatives. The method is called Lie symmetry method after the name of Sophus Lie. In this method, the transformations are found with the help of Lie invariance condition. The solutions of Lie invariance condition are known as infinitesimals or transformations. In the present work, the method is applied on Riccati equation. The method is then explained with the help of examples.

II. LITERATUREREVIEW

Olver and Rosenau (1987) studied group invariant solutions of differential equations.Invariant solutions of an ordinary differential equation using infinitesimals of an admitted Lie group of transformations were found by **Bluman (1990)**.Similarity solutions of second-order partial differential equations with two independent and one dependent variableswere studied by **Dresner (1988).Clarkson and Olver (1996)**reduced the order of an ordinary differential equation by three in case when the equation has a symmetric group. A complete symmetry analysis of the one dimensional Black-Scholes model is conducted by **Gazizov and Ibragimov (1998)**.**Leach and Bouquet (2002)** demonstrated that integrating factor of an ODE is equivalent to the existence of suitable Lie symmetries.**Manjit Singh (2015)** reduced the order of ordinary differential equations using Lie group of transformations. He showed that for an ordinary differential equation admitting one parameter Lie group symmetry, order of differential equation, in principle, can always be reduce by one.**Kumar G. (2019)** used Lie symmetry method to find general solution of homogeneous ODE of first order. First integrals of a second order ODE is studied by **Hu and Du (2019)**.**Kumar G**. (2020) used Lie invariance condition to find symmetries for homogeneous and linear ODE of first order.

III. LIE SYMMETRY METHOD

Lie symmetry method for first order ordinary differential equation is presented in this section. Then the method is applied on Riccati equation. Lie invariance condition is obtained for standard form of Riccati equation.
Let usconsidera first order ODE given by–

$$\frac{dx}{dt} = F(t, x), \tag{1}$$

where x dependent and t is independent variable.
We considerLie group inone parameter as

$$\bar{t} = f(t, x; \varepsilon), \ \bar{x} = g(t, x; \varepsilon). \tag{2}$$

If(1) isinvariant under (2), then we have –

$$\frac{d\bar{x}}{d\bar{t}} = F(\bar{t}, \bar{x}), \tag{3}$$

Solution of (3) gives -

$$\frac{g_t + g_x F(t,x)}{f_t + f_x F(t,x)} = F\left(f(t,x,\varepsilon), g(t,x,\varepsilon) \right). \tag{4}$$

Now infinitesimals/transformations,*X* and *T*, to transform (1) to canonical form are given by the solutions of Lie's invariance condition-

$$X_t + (X_x - T_t)F - T_x F^2 = TF_t + XF_x \tag{5}$$

To solve (5) we assume some form of one infinitesimal, say *T*. The other infinitesimal *X*is then foundby Lagrange's method through auxiliary equations. From*X* and *T*, we transform (1) to canonical coordinates *(r,s)*using-

$$r_t T + r_x X = 0 \tag{6}$$

$$s_t T + s_x X = 1 \tag{7}$$

We solve (6) and (7) using Lagrange's method of characteristics to find *r* and *s*.We thenfound -

$$\frac{ds}{dr} = \frac{s_t + s_x \dfrac{dx}{dt}}{r_t + r_x \dfrac{dx}{dt}} \tag{8}$$

Equation (8) issolved to get general solution of equation (1).

Now we apply this method to Riccati equation-

$$\frac{dx}{dt} = P(t)x^2 + Q(t)\,x + R(t) \tag{9}$$

Here $F = P(t)x^2 + Q(t)\,x + R(t)$

Lie invariance condition in this case is given by –

$$\begin{aligned} &X_t + (X_x - T_t)[P(t)x^2 + Q(t)\,x + R(t)] - T_x[P(t)x^2 + Q(t)\,x + R(t)]^2 \\ &= T[P(t)x^2 + Q(t)\,x + R(t)]_t + X[P(t)x^2 + Q(t)\,x + R(t)]_x \end{aligned} \tag{10}$$

We take $T = 0$, then (10) becomes –

$$X_t + X_x[P(t)x^2 + Q(t)\,x + R(t)] = X[P(t)x^2 + Q(t)\,x + R(t)]_x \tag{11}$$

Let x_1 be one solution of equation (1) then we have -

$$X = (x - x_1)^2 F(t)$$

Here *F* satisfies –

$$F' + (2Px_1 + Q)F = 0 \tag{12}$$

By canonical variable

From (6) we have –

$$r_t 0 + r_x (x - x_1)^2 F(t) = 0$$

$$r_x (x - x_1)^2 F(t) = 0 \tag{13}$$

From (7) we have –

$$s_t 0 + s_x (x - x_1)^2 F(t) = 1$$

$$s_x (x - x_1)^2 F(t) = 1 \tag{14}$$

Now by (13), we have -

$$\frac{dr}{0} = \frac{dt}{0} = \frac{dx}{(x - x_1)^2 F(t)}$$

(i) (ii) (iii)

From (i) & (ii)

$r = t = c, \quad say$

Now by (14), we have -

$$\frac{ds}{1} = \frac{dt}{0} = \frac{dx}{(x - x_1)^2 F(t)}$$

(i) (ii) (iii)

From (i) and (iii), we have -

$$\int ds = \int \frac{dx}{(x - x_1)^2 F(t)}$$

$$s = \frac{(x - x_1)^{-1}}{-1 F(t)} + S$$

$$s = S - \frac{1}{(x - x_1)F}$$

Take $S = 0$, we get -

$$s = -\frac{1}{(x - x_1)F}$$

$$(x - x_1)\, F = -\frac{1}{s}$$

$$(x - x_1) = -\frac{1}{sF}$$

$$x = x_1 - \frac{1}{sF}$$

Now we solve -

$$\frac{ds}{dr} = \frac{s_t + s_x \dfrac{dx}{dt}}{r_t + r_x \dfrac{dx}{dt}} \tag{15}$$

to find general solution of (9).

IV. EXAMPLES

In this section, Lie symmetry method for Riccati equation is explained with the help of examples.

Example 4.1 Consider following Riccati equation

$$\frac{dx}{dt} = \frac{1}{t}x^2 + \frac{1}{t}x - \frac{2}{t} \tag{16}$$

Compare the equation (1) with general form of Riccati Equation.

$$\frac{dx}{dt} = P(t)x^2 + Q(t)x + R(t) \tag{17}$$

Here $P(t) = \frac{1}{t},\ Q(t) = \frac{1}{t}, R(t) = \frac{-2}{t}$ and

$$F = \frac{1}{t}x^2 + \frac{1}{t}x - \frac{2}{t}$$

Lie invariance condition is given by –

$$X_t + X_x[P(t)x^2 + Q(t)\ x + R(t)] = X[P(t)x^2 + Q(t)\ x + R(t)]_x \tag{18}$$

One solution of equation (16) is $x_1 = 1$

Here

$$X = (x - x_1)^2 F(t)$$

$$X = (x - 1)^2 F(t) \tag{19}$$

From (12), we have –

$$F' + (2Px_1 + Q)F = 0$$

Which gives –

$$F'(t) + \left(\frac{2}{t} + \frac{1}{t}\right)F = 0$$

$$F^{|}(t) = -\frac{3}{t}F$$

$$\frac{dF}{dt} = \frac{-3}{t}F$$

$$\int \frac{dF}{F} = -3\int \frac{dt}{t}$$

$$\log_e F = -3\log t + \log C$$

$$F = Ct^{-3}$$

We take *C=1*

Then $F = t^{-3}$ (20)

So that $X = (x-1)^2 t^{-3}$ is the solution of Lie invariance condition (18)

Now from (13), we have -

$$r_x (x-1)^2 t^{-3} = 0$$

Using Lagrange's method of characteristics, we get auxiliary equations as -

$$\frac{dt}{0} = \frac{dx}{(x-1)^2 t^{-3}} = \frac{dr}{0}$$

(i) (ii) (iii)

From (i) and (iii), we get –

$r = t = c, \quad say$ (21)

From (14), we get –

$$s_x (x-1)^2 t^{-3} = 1$$

Using Lagrange's method of characteristics, we get auxiliary equations as -

$$\frac{dt}{0} = \frac{dx}{(x-1)^2 t^{-3}} = \frac{ds}{1}$$

(i) (ii) (iii)

From (ii) and (iii), we get –

$$ds = \frac{dx}{(x-1)^2 t^{-3}}$$

$$ds = (x-1)^{-2} t^3 dx$$

Put t = c, we get –

$$ds = (x-1)^{-2} c^3 dx$$

$$\int ds = c^3 \int (x-1)^{-2} dx$$

$$s = c^3 \frac{(x-1)^{-1}}{-1}$$

$$s = \frac{c^3}{1-x}$$

Since $c = t$, we get -

$s = \frac{t^3}{1-x}$ (22)

By canonical form -

$$\frac{ds}{dr} = \frac{3t^2 - 3t^2 + t^3 x^1}{(1-x)^2} \tag{23}$$

From $1 - x = \frac{t^3}{s}$, we get –

$$x = 1 - \frac{t^3}{s}$$

Put $t = r$, we get -

$$x = 1 - \frac{r^3}{s}$$

From (23), we have -

$$\frac{ds}{dr} = \frac{3r^2 - 3r^2\left[1\frac{r^3}{s}\right] + r^3\left[\frac{1}{r}x^2 + \frac{1}{r}x - \frac{2}{3}\right]}{\left(\frac{r^3}{s}\right)^2}$$

$$= \frac{3r^2 - 3r^2 + \frac{3r^5}{s} + r^3\left[\frac{1}{r}\left(1 - \frac{r^3}{s}\right)^2 + \frac{1}{r}\left(1 - \frac{r^3}{s}\right) - \frac{2}{r}\right]}{\frac{r^6}{s^2}}$$

$$= \frac{\frac{3r^5}{s} + r^3\left[\frac{1}{r}\left(1 + \frac{r^6}{s^2} - \frac{2r^3}{s}\right) + \left(\frac{1}{r} - \frac{r^2}{s} - \frac{2}{r}\right)\right]}{\frac{r^6}{s^2}}$$

$$\frac{ds}{dr} = \frac{\left[\frac{3r^5}{5} + r^2 + \frac{r^8}{s^2} - \frac{2r^5}{s} + r^2 - \frac{r^5}{s} - 2r^2\right]}{\frac{r^6}{s^2}}$$

$$\frac{ds}{dr} = \frac{r^8}{s^2} \times \frac{s^2}{r^6}$$

$$\frac{ds}{dr} = r^2$$

$$\int ds = \int r^2 \, dr$$

$$s = \frac{r^3}{3} + C$$

Putting value of s, we get -

$$\frac{r^3}{1-x} = \frac{r^3}{3} + C$$

$$\frac{r^3}{1-x} = \frac{r^3 + 3C}{3}$$

$$1 - x = \frac{3r^3}{r^3 + 3C}$$

$$x = 1 - \frac{3r^3}{r^3 + 3C}$$

$$x = 1 - \frac{3t^3}{t^3 + 3C} \tag{24}$$

(24) gives the general solution of Riccati equation (16).

Example 4.2 Consider following Riccati equation

$$\frac{dx}{dt} = x^2 - 1 \tag{25}$$

Compare the equation (25) with general form of Riccati Equation.

$$\frac{dx}{dt} = P(t)x^2 + Q(t)x + R(t) \tag{26}$$

Here $P(t) = 1,\ Q(t) = 0,\ R(t) = -1$ and

$$F = x^2 - 1$$

One solution of equation (25) is $x_1 = 1$

Lie invariance condition is given by –

$$X_t + X_x[P(t)x^2 + Q(t)\ x + R(t)] = X[P(t)x^2 + Q(t)\ x + R(t)]_x \tag{27}$$

Here

$$X = (x - x_1)^2 F(t)$$

$$X = (x - 1)^2 F(t)$$

From (12), we have –

$$F' + (2Px_1 + Q)F = 0$$

Which gives –

$$F'(t) + 2F = 0$$

$$F'(t) = -2F$$

$$\int \frac{dF}{F} = -\int 2\,dt$$

$$\log_e F = -2\log t + \log C$$

$$F = Ce^{-2t}$$

We take *C =1*

$$F = e^{-2t} \tag{28}$$

So we get –

$$X = (x - 1)^2 e^{-2t} \tag{29}$$

as solution of Lie invariance condition.

Now from (13), we have -

$$r_x (x - 1)^2 e^{-2t} = 0$$

Using Lagrange's method of characteristics, we get auxiliary equations as -

$$\frac{dt}{0} = \frac{dx}{(x-1)^2 e^{-2t}} = \frac{dr}{0}$$

(i) (ii) (iii)

From (i) and (iii), we get –

$$r = t = c, \quad say \tag{30}$$

From (14), we get –

$$s_x (x - 1)^2 e^{-2t} = 1$$

$$\frac{dt}{0} = \frac{dx}{(x-1)^2} e^{-2t} = \frac{ds}{1}$$

(i) (ii) (iii)

From (ii) and (iii), we get -

$$ds = \frac{dx}{(x-1)^2 e^{-2t}}$$

$$ds = \frac{e^{2t}}{(x-1)^2} dx$$

Put $t = c$, we get -

$$\int ds = \int e^{2c} (x-1)^{-2} dx$$

$s = - e^{2c} (x-1)^{-1}$

$$s = -\frac{e^{2c}}{(x-1)}$$

$$s = \frac{e^{2c}}{(1-x)} \tag{31}$$

So the canonical coordinates are $(r, s) = \left(t, \frac{e^{2t}}{(1-t)} \right)$

By canonical from -

$$\frac{ds}{dr} = \frac{D_t s}{D_t r} = \frac{2(1-x)e^{2t} - e^{2t}(-x^1)}{\frac{(1-x)^2}{1}}$$

$$= \frac{2e^{2t} - 2xe^{2t} - e^{2t}(-x^2+1)}{(1-x)^2}$$

$$= \frac{2e^{2t} - 2xe^{2t} + e^{2t}x^2 - e^{2t}}{(1-x)^2}$$

$$\frac{ds}{dr} = \frac{e^{2t} - 2xe^{2t} + e^{2t}x^2}{(1-x)^2} \tag{32}$$

Now we have-

$$s = \frac{e^{2t}}{1-x}$$

So that

$1 - x = \frac{e^{2t}}{s}$ which gives -

$$x = 1 - \frac{e^{2r}}{s}$$

From (32), we get –

$$\frac{ds}{dr} = \frac{e^{2r} - 2xe^{2r} + e^{2r}x^2}{\left(\frac{e^{2r}}{s}\right)^2}$$

$$= \frac{e^{2r} - 2e^{2r}\left[1 - \frac{e^{2r}}{s}\right] + e^{2r}\left[1 - \frac{e^{2r}}{s}\right]^2}{\frac{e^{4r}}{s^2}}$$

$$= \frac{e^{2r} - 2e^{2r} + \frac{2e^{4r}}{s} + e^{2r}\left[1 + \frac{e^{4r}}{s^2} - f\frac{e^{2r}}{s}\right]}{\frac{e^{4r}}{s^2}}$$

$$= \frac{e^{2r} + \frac{2e^{4r}}{s} + e^{2r} + \frac{e^{6r}}{s^2} - \frac{2e^{4r}}{s}}{\frac{e^{4r}}{s^2}}$$

$$\frac{ds}{dr} = \frac{e^{6r}}{s^2} \times \frac{s^2}{e^{4r}}$$

$$\frac{ds}{dr} = e^{2r}$$

$$\int ds = \int e^{2r}\,dr$$

$$s = \frac{e^{2r}}{2} + c$$

Put values of *s*, we get -

$$\frac{e^{2r}}{(1-x)} = \frac{e^{2r}}{2} + c$$

$$\frac{e^{2r}}{(1-x)} = \frac{e^{2r} + 2c}{2}$$

$(1–x)\,(e^{2r} + 2c) = 2e^{2r}$

$$(1-x) = \frac{2e^{2r}}{e^{2r} + 2c}$$

$$x = 1 - \frac{2e^{2t}}{e^{2t} + 2c} \quad (33)$$

(33) give the general solution of Riccati equation (25)

V. CONCLUSION

Lie symmetry method has an advantage over other methods as it solves differential equations based on algorithm. The method can be sued for any order ODE. It reduces the order of ordinary/partial differential equation with the help of infinitesimals/transformations. These transformations exhibit group properties and help in integration of the differential equation.In this work, this method is explained for Riccati differential equation. Then it is used on some of the equations to find their general solutions. It is found that Lie symmetry method gives exact solution of Riccati equation.

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