

Spectral Dynamic Analysis of Torsional Vibrations of Axially Exponentially Graded Thin-Walled Open Section Beams Rotationally Restrained at One End and Transversely Restrained at the Other End

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Abstract: *The present paper deals with spectral dynamic analysis of free torsional vibration of doubly symmetric thin-walled beams of open section. Spectral frequency equation is derived in this paper for the case of rotationally restrained doubly-symmetric thin-walled beam with one end rotationally restrained and transversely restrained at the other end. The resulting transcendental frequency equation with appropriate boundary conditions is derived and is solved for varying values of warping parameter and the rotational and transverse restraint parameter. The influence of rotational restraint parameter, transverse restraint parameter and warping parameter on the free torsional vibration frequencies is investigated in detail. A MATLAB computer program is developed to solve the spectral frequency equation derived in this paper. Numerical results for natural frequencies for various values of rotational and transverse restraint parameters for various values of warping parameter are obtained and presented in both tabular as well as graphical form showing the influence of these parameters on the first fundamental torsional frequency parameter.*

Keywords: *Thin-walled beam, open section, torsion, spectral dynamic analysis, restrained cantilever.*

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I. INTRODUCTION

It is very well known that in practical situations, the boundary conditions of structural members will be quite complex and can be simulated by using translational springs and rotational springs with appropriate combinations of the same. There exist a good number of research efforts in this direction and many researchers have addressed this problem of vibrations of generally restrained beams with various combinations of boundary conditions [1-25]. The combined effect of rotary inertia, shear deformation and root flexibility has been investigated experimentally by Beglinger et al. [12]. A considerable amount of theoretical work has been done in the field of vibration dealing with the computation of natural frequencies and mode shapes of cantilever beams with flexible roots [6, 9, 17, 19, 20, 22]. Kameswara Rao [23] presented a closed form equation for computing fundamental frequency of cantilever blade taking into account the resilience of the clamped end. Experimental verification of the results for this case was also carried out by Abbas and Irretier [24]. Kameswara Rao and Mirza [25] derived the transcendental frequency equation and mode shape expressions for the case of generally restrained Euler-Bernoulli beams and presented extensive numerical results for various values of linear and rotational restraint parameters.

While there are a number of publications on flexural vibrations of elastically restrained cantilever beams, the literature on torsional vibrations of doubly symmetric thin-walled beams of open section is rather rare. Free torsional vibrations and stability of doubly-symmetric long thin-walled beams of open section were investigated by Gere [26] and Christiano and Salmela [27]. Numerical values of exact torsional natural frequencies of beams with circular cross-section, where nonuniform warping does not arise, were presented by Gorman [26] and Belvins [27] for different classical boundary conditions. Torsional vibration frequencies for beams of open thin-walled sections, subjected to several combinations of classical boundary conditions, taking into account warping effects were first derived by Gere [28]. Including elastic torsional and warping restraints, Carr [29] and Christino and Salmela [30] presented numerical results using approximate methods for the calculation of natural frequencies. For the case of torsional frequencies of circular shafts and piping with elastically restrained edges, Kameswara Rao [31] derived exact frequency equation and presented corresponding numerical results for a wide range of non-dimensional parameters.

It can be seen from the very recent review presented by Sapountzakis [32] that the problem of free torsional vibration analysis of doubly-symmetric thin-walled I-beams or Z-beams subjected to partial warping

restraint is not being addressed till now in the available literature. In view of the same, an attempt has been made in this paper to present a spectral dynamic analysis of free torsional vibration of doubly-symmetric thin-walled beams of open section with one end rotationally restrained and the other transversely restrained including the effects of warping parameter. Spectral frequency equation is derived for this case and the resulting transcendental frequency equation is solved for varying values of warping parameter and the partial restraint parameters. The influence of rotational and transverse restraint parameter along with warping parameter on the free torsional vibration frequencies is investigated in detail by utilising a Matlab computer program developed especially to solve the spectral frequency equation derived in this paper. Numerical results for natural frequencies for various values of partial rotational and transverse restraint parameters are obtained and presented in both tabular as well as graphical form for use in design, showing their parametric influence clearly.

II. FORMULATION AND ANALYSIS

Consider a long doubly-symmetric thin-walled beam of open cross section of length L and the beam as undergoing free torsional vibrations. The corresponding differential equation of motion can be written as:

$$e^{2\beta l} \left[\left(\frac{\partial^4 \varphi}{\partial x^4} + 4(\beta l) \frac{\partial^3 \varphi}{\partial x^3} + 4(\beta l)^2 \frac{\partial^2 \varphi}{\partial x^2} \right) + K^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + 2(\beta l) \frac{\partial \varphi}{\partial x} \right) \right] + \lambda^2 \varphi = 0 \quad (1)$$

where,

E= young's modulus, C_w =warping constant, G =shear modulus, C_s = torsion constant, ρ =mass density of the material of the beam, I_p =polar moment of inertia, φ = angle of twist, z = distance along the length of the beam. For free torsional vibrations, the angle of twist $\varphi(z, t)$ can be expressed in the form.

$$\begin{aligned} \varphi(z, t) &= x(z)e^{i\omega t} \\ x(z) &= Ce^{mz} \end{aligned} \quad (2a) \quad (2b)$$

In which $x(z)$ is the modal shape function corresponding to each beam torsional natural frequency ω .

The expression for $x(z)$ which satisfies Eqn. (1) can be written as:

$$x(z) = Ae^{+\alpha z} + Be^{-\alpha z} + Ce^{+i\beta z} + De^{-i\beta z} \quad (3)$$

in which,

$$\beta L, \alpha L = \sqrt{\frac{\mp K^2 + \sqrt{K^4 + 4\lambda^2}}{2}} \quad (4)$$

where,

$$K^2 = \left(\frac{GC_s L^2}{EC_w} \right); \quad \text{Non- dimensional warping parameter}$$

$$\lambda^2 = \left(\frac{\rho I_p \omega^2 L^4}{EC_w} \right); \quad \text{Non- dimensional frequency parameter}$$

From Eqn. (4), we have the following relation between αL and βL

$$(\alpha L)^2 = (\beta L)^2 + K^2 \quad (5)$$

Knowing α and β , the frequency parameter λ can be evaluated using the following equation:

$$\lambda^2 = (\alpha L)(\beta L) \quad (6)$$

The four arbitrary constants A, B, C and D in Eqn. (3) can be determined from the boundary conditions of the beam. For any single-span beam, there will be two boundary conditions at each end and these four conditions then determine the corresponding frequency and mode shape expressions.

III. DERIVATION OF SPECTRAL FREQUENCY EQUATION

Consider a thin-walled doubly symmetric I-beam with one end rotationally restrained and the other end transversely restrained as shown in figure 1, undergoing free torsional vibrations. In order to derive the spectral frequency equation for this case, let us first introduce the related nomenclature.

The variation of angle of twist φ with respect to z is denoted by $\theta(z)$. The flange bending moment and the total twisting moment are given by $M(z)$ and $T(z)$. Considering clockwise rotations and moments to be positive, we have

$$\theta(z) = \frac{d\varphi}{dz}; \quad hM(z) = -EC_w \frac{d^2 \varphi}{dz^2} \quad (7)$$

$$T(z) = -EC_w \frac{d^3 \varphi}{dz^3} + GC_s \frac{d\varphi}{dz} \quad (8)$$

where $EC_w = \frac{I_f h^2}{2}$. I_f being the flange moment of inertia and h is the distance between the center lines of the flanges of a thin-walled I-beam.

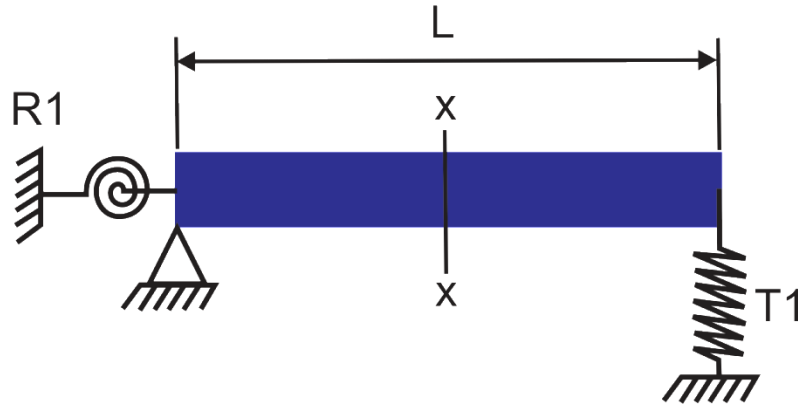


Fig.1 (a) A Thin-Walled Open Section Beam Rotationally Restrained at One End and Transversely Restrained at the Other End

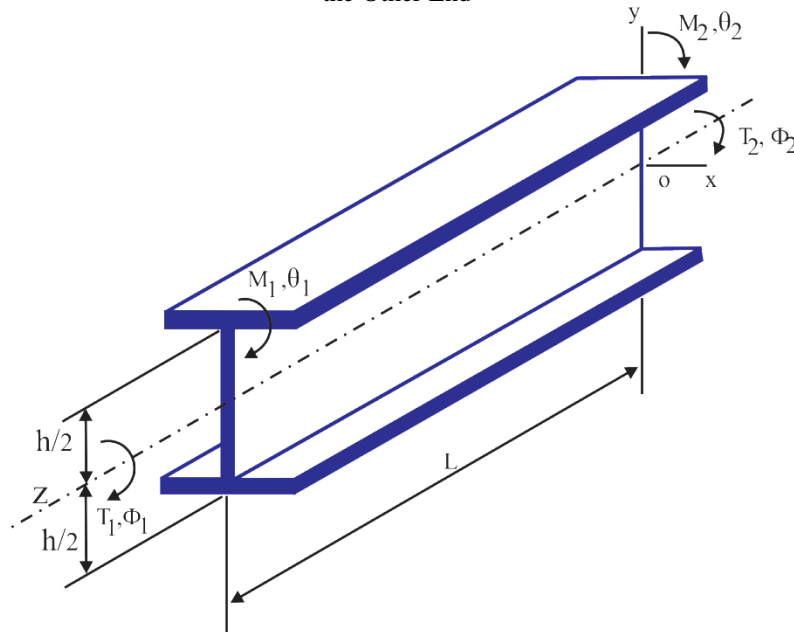


Fig. 1 (b) Cross-section of the beam at x-x

Taking S as the stiffness of the rotational spring and $R = (SL/EC_W)$ as the non-dimensional rotational spring stiffness parameter and $Z=(z/L)$ as the non-dimensional length of the beam, the boundary conditions can be easily identified as follows:

$$\text{At } Z=0, \varphi = 0, \frac{d^2\varphi}{dz^2} = R \frac{d\varphi}{dz} \quad (9)$$

$$\text{And at } Z=L, \frac{d^3\varphi}{dz^3} - K^2 \frac{d\varphi}{dz} = T\varphi, \frac{d^2\varphi}{dz^2} = 0 \quad (10)$$

The spectral frequency equation obtained is as given below:

$$RS_1 + F_1S_2 + TF_2Q_{1m2} + RTF_3S_3 = 0 \quad (11)$$

where

$$F_1 = \frac{(\alpha^2 + \beta^2)}{(\alpha^2 \beta^2)}; F_2 = \frac{(\alpha^4 + 2\alpha^2 \beta^2 + \beta^4)}{(\alpha^3 \beta^3)}; F_3 = \frac{(\alpha^2 + \beta^2)}{(\alpha^3 \beta^3)} \quad (12)$$

$$Q_1 = \frac{1 + e^{2L(\alpha + i\beta)}}{4e^{L(\alpha + i\beta)}}; Q_2 = \frac{1 + e^{2L(\alpha - i\beta)}}{4e^{L(\alpha - i\beta)}} \quad (13)$$

$$Q_{1m2} = (Q_1 - Q_2) \quad (14)$$

$$S_1 = (F_1 Q_{1p2} + F_2 Q_{1m2} + 2); S_2 = (\alpha^3 Q_{3p4} - \beta^3 Q_{3m4}); S_3 = (\alpha Q_{3m4} - \beta Q_{3p4}) \quad (15)$$

Four degenerate cases spectral frequency equations can be easily obtained from Equation (11) as follows:

Case (i). For $R = 0$ and $T = \infty$, we get the case of simply-supported beam for which we obtain

$$Q_{1m2} = 0 \quad (16)$$

Case (ii). For $R = \infty$ and $T = 0$, we get the case of cantilever beam with restrained warping for which we obtain

$$S_1 = 0 \quad (17)$$

Case (iii). For $R = 0$ and $T = 0$, we get the case of cantilever beam with unrestrained warping for which we obtain

$$S_2 = 0 \quad (18)$$

Case (iv). For $R = \infty$ and $T = \infty$, we get the clamped-simply supported beam case for which we obtain

$$S_3 = 0 \quad (19)$$

IV. RESULTS AND DISCUSSIONS

Numerical results for the first three natural torsional frequencies of vibration of thin-walled beams of open section are obtained by solving the transcendental spectral frequency Eq. (11) using trial-and-error method. The Muller's iteration technique based on bisection method is coded in Matlab and the same is utilised in generating the numerical data and the same is presented in several tables and graphs for use in design.

It should be mentioned here that even though several studies are made by researchers in the area of torsional frequencies of thin-walled beams of open section, numerical values are not made available for use in design. As is known, graphical results can help us only in understanding the trend of variation of natural frequencies due to the increase in warping parameter K and the partial warping restraint parameters R and T , but will not provide the frequencies to the four digit accuracy which we require for using the same for design.

For the case of cantilever thin-walled beam with partially restrained warping (R), R varying from 0 to 10^{+17} at the left end and with partial linear transverse restraint (T), T varying from 0 to 10^{+17} at the other, the fundamental mode torsional frequencies for a fixed value of warping parameter $K=0.0$ are presented in Table 1. The fundamental mode torsional frequencies are determined for a wide range of R and T but only a few are presented in Table.1. Figure 2 represents the variation of frequency parameter with warping parameter ($K = 0$ to 10) for $R = 0$ and $R = 10^{+17}$. Whereas, Figure 3 is drawn to clearly show the variation of the fundamental first mode frequencies with varying values of K and R . It is observed that for a given value of R , the frequency parameter increases with increase in warping parameter value K .

From figures 2 and 3, we can easily see that the increase in warping parameter K is to increase the fundamental mode torsional frequencies significantly. For values of K greater than 10, we can easily notice that the frequencies of cantilever with unrestrained as well as completely restrained thin-walled beams almost tend to converge to a constant value as K approaches higher values such as 80. For a constant value of warping parameter K , the increase in values of partial warping parameter R from 0 to infinity (10^{18}) results in consistent increase in the values of fundamental mode frequencies as the cantilever end becomes stiffer and stiffer.

From the definition of non-dimensional warping parameter K (4b), we can understand that the torsional frequency increases for increasing values of torsion constant C_s or decreasing values of warping constant C_w . Effect of K also can be seen to be more predominant compared to the effect of partial warping restraint K . This can be seen from Figure 3 whereas K is increasing from 0 to 80, the two curves related to cantilever beam fully restrained and the one with unrestrained warping are almost converging to the same value and hence we can conclude that the boundary condition has insignificant effect on the natural torsional frequencies of thin-walled doubly symmetric beams for very high values of warping parameter K .

Fundamental mode torsional frequencies of thin-walled beams for wide range of values of warping parameter K from 1 to 80 and the partial warping restraint R from 0 to 10^{18} are calculated. These results are also plotted in Figures 4 to 5 showing clearly the variation of fundamental natural torsional frequency with varying values of warping parameter K and the partial warping restraint R .

The percentage variation of frequency parameter with increasing values of R , as K varies from 0 to 80 is presented in Figure. 6. The percentage variation of frequency parameter changes from 96.29025 to 83.37943 when the value of K varies from 0 to 80. Similarly, the percentage variation of frequency parameter with K as R varies from 0.01 to 10^{+18} is presented in Figure. 7. The percentage variation of frequency parameter changes from 77.81 to 0.62 when the value of R varies from 0.01 to 10^{+18} .

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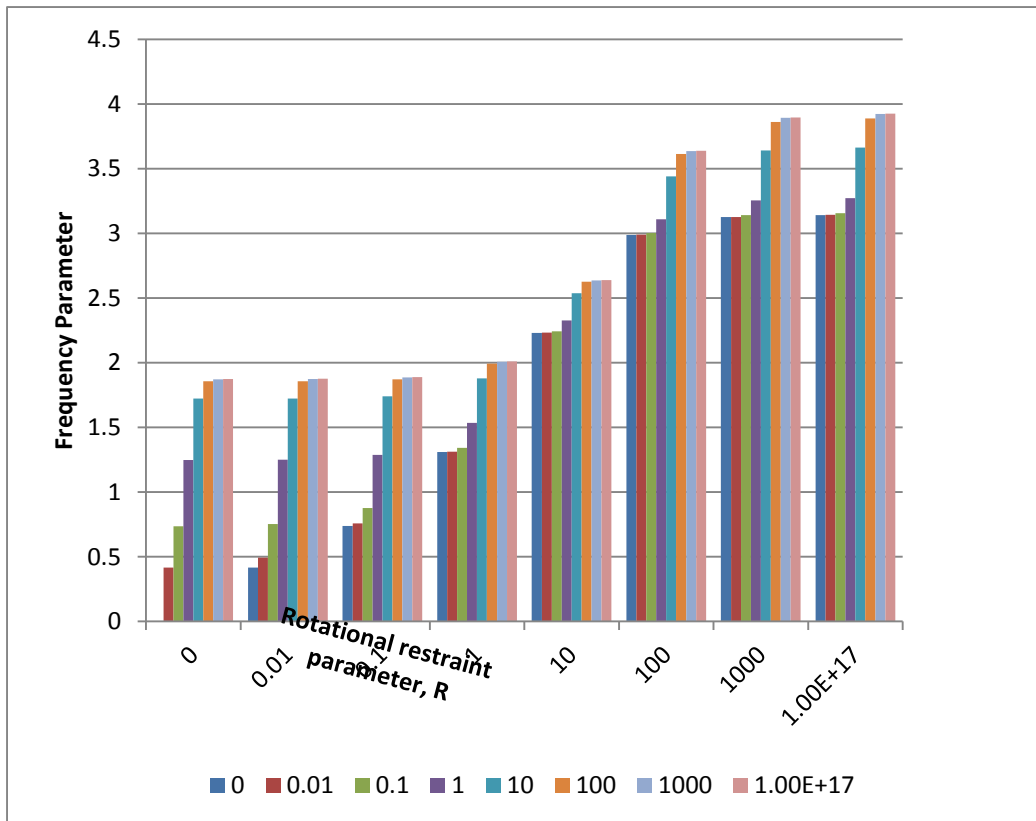
The values of second and third mode torsional natural frequencies of thin-walled beams of open section for various values of warping parameter K from 0.01 to 200 and partial warping restraint parameter R from 0.01 to 1000 in Tables 2 and 3 respectively. These numerical values are plotted in Figures 8 and 9 for the second mode and Figures 10 and 11 for the third mode, showing clearly the influence of warping parameter K and partial warping restraint R on the non-dimensional natural frequency parameter λ .

The authors sincerely hope that this detailed data presented in this paper will be quite useful in design of such systems and also to establish accuracy of frequencies obtained by using latest approximate methods such as Generalised Differential Quadrature Method (GDQM), Differential Transform Method (DTM), Adomian Decomposition Method (ADM).or any other method such as Finite Element Method.

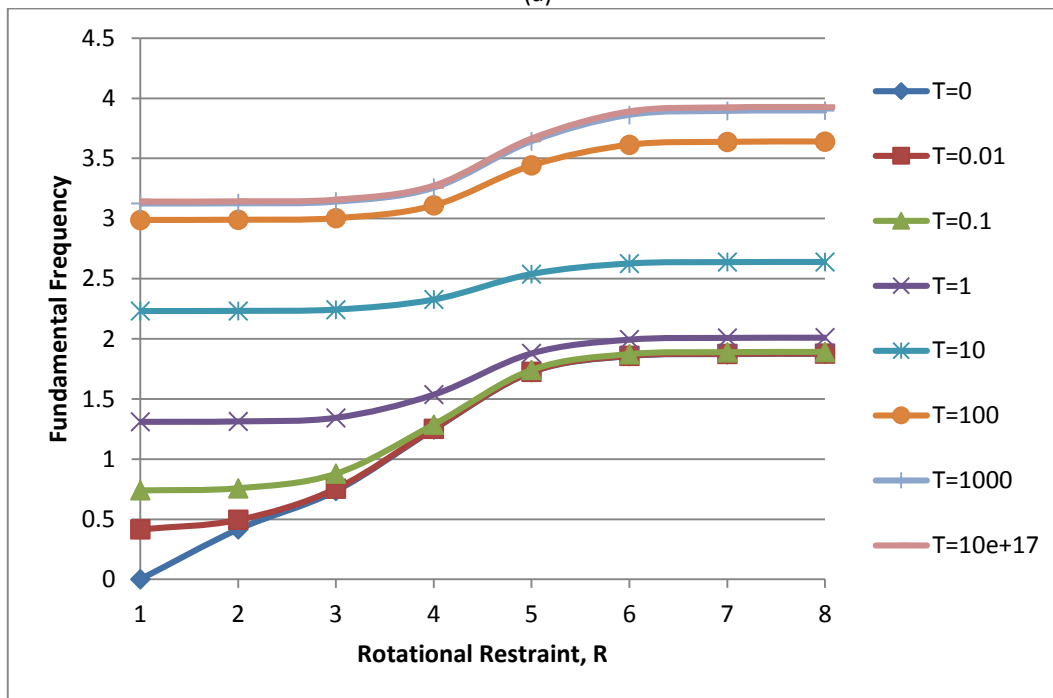
Spectral dynamic analysis of free torsional vibration of doubly symmetric thin-walled beams of open section is carried out and detailed results of this study are presented in this paper suitable for use in design and also for checking approximate solutions obtained for their accuracy. For the case of a cantilever thin-walled beam of doubly symmetric open cross-section undergoing free torsional vibrations and subjected to partial warping restraint, the spectral frequency equation is derived in this paper. The resulting transcendental frequency equation for the case of cantilever boundary conditions is solved for thin-walled beams of open cross section for varying values of warping parameter and the partial warping restraint parameter. The influence of partial warping restraint parameter R and the warping parameter K on the free torsional vibration frequencies is investigated in detail and significant amount of numerical frequency data is generated. Using a MATLAB computer program developed to solve the spectral frequency equation derived, numerical results for the first three modes of torsional natural frequencies for various values of rotational restraint parameter R and warping restraint parameter K are obtained and are presented in both tabular as well as graphical form showing their parametric influence clearly. In comparison with the partial warping restraint parameter R , the warping parameter K is found to have significant effect on the torsional natural frequencies not only of the fundamental mode but also of higher modes.

Table 1. First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 0$

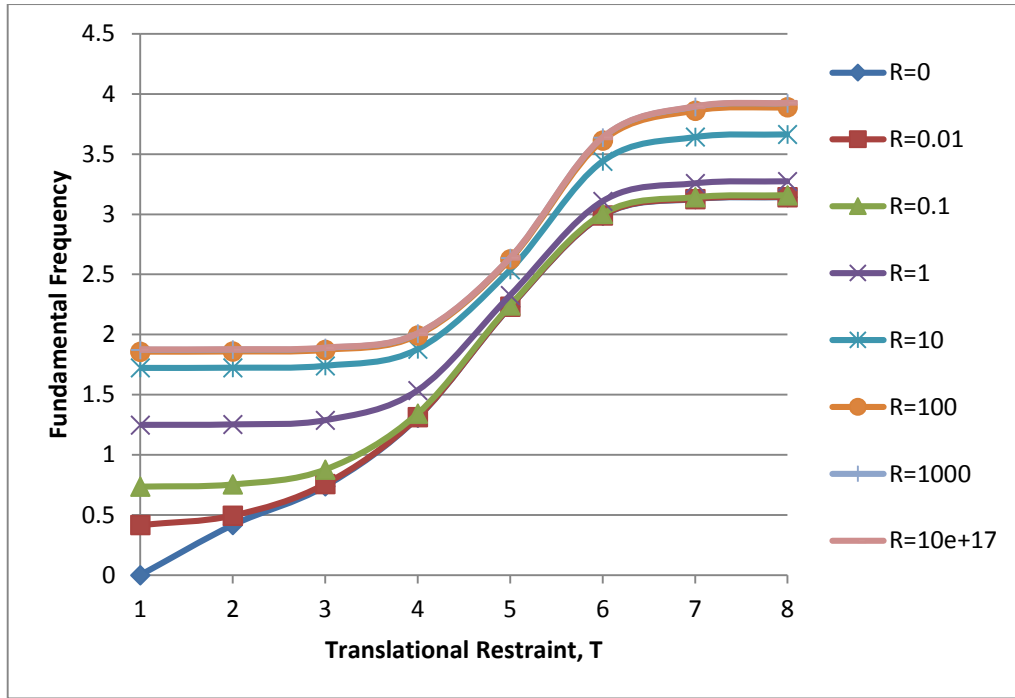
| R | $T = 0$ | $T = 0.01$ | $T = 0.1$ | $T = 1$ | $T = 10$ | $T = 100$ | $T = 1000$ | $T = 10^{17}$ |
|-----------|---------|------------|-----------|---------|----------|-----------|------------|---------------|
| 0 | 0 | 0.41616 | 0.73973 | 1.30981 | 2.23133 | 2.98864 | 3.12608 | 3.14159 |
| 0.01 | 0.41595 | 0.49481 | 0.75769 | 1.31339 | 2.23256 | 2.99011 | 3.12766 | 3.14318 |
| 0.1 | 0.73578 | 0.75406 | 0.87821 | 1.34368 | 2.24336 | 3.00301 | 3.14155 | 3.15718 |
| 1 | 1.24792 | 1.25200 | 1.28704 | 1.53581 | 2.32647 | 3.10846 | 3.25665 | 3.27329 |
| 10 | 1.72274 | 1.72455 | 1.74058 | 1.87929 | 2.53883 | 3.44122 | 3.64228 | 3.66464 |
| 100 | 1.85679 | 1.85833 | 1.87205 | 1.99395 | 2.62616 | 3.61335 | 3.86146 | 3.88919 |
| 1000 | 1.87323 | 1.87475 | 1.88824 | 2.00836 | 2.63761 | 3.63772 | 3.39401 | 3.92269 |
| 10^{17} | 1.87510 | 1.87662 | 1.89008 | 2.01000 | 2.63893 | 3.64054 | 3.89780 | 3.92660 |



(a)



(b)

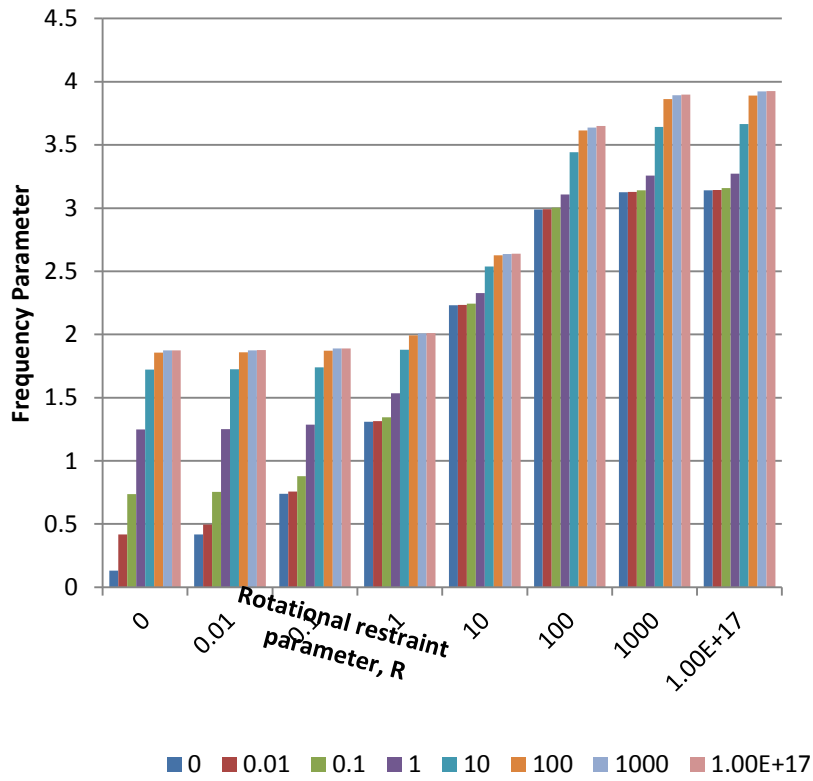


(c)

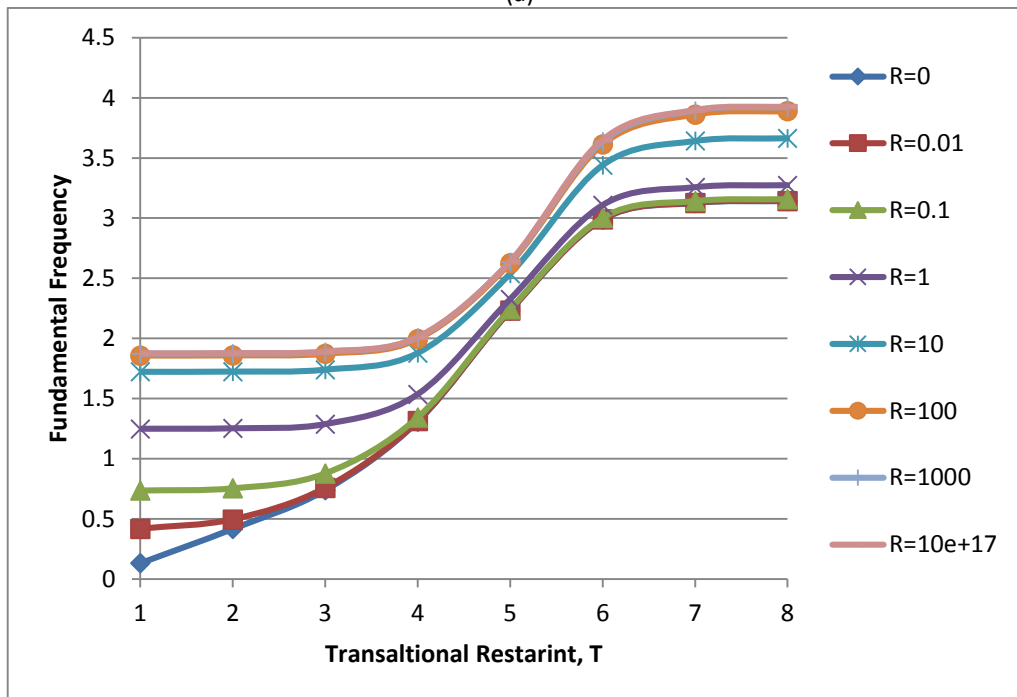
Fig. 2 (a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter ($K=0$).

Table.2 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 0.01$

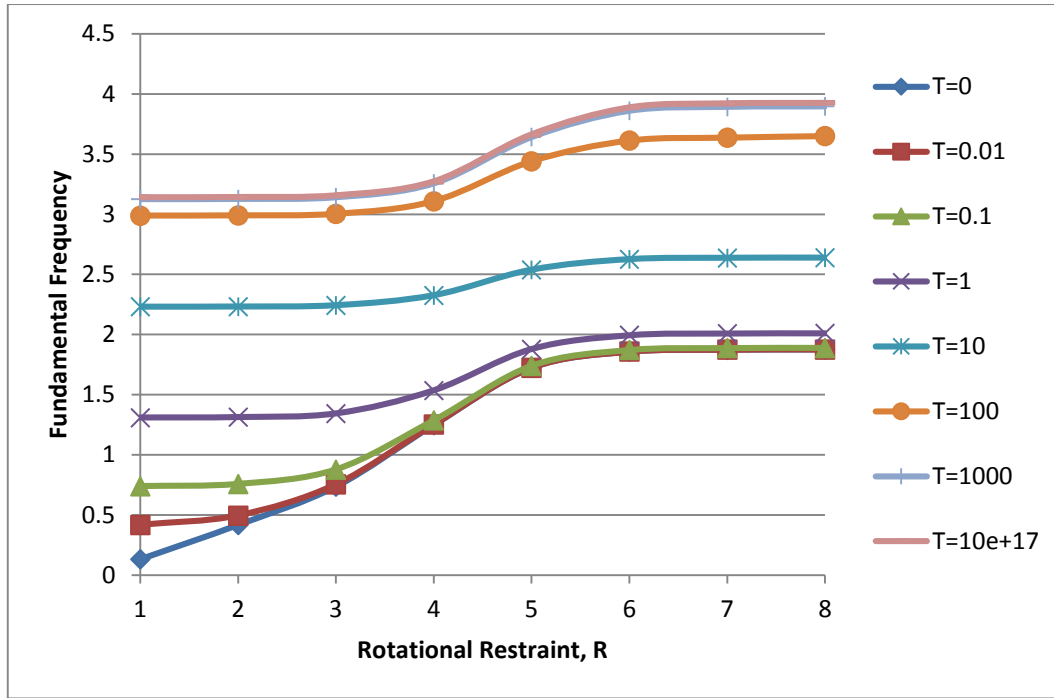
| R | T = 0 | T = 0.01 | T = 0.1 | T = 1 | T = 10 | T = 100 | T = 1000 | T = 10^{17} |
|-----------|--------|----------|---------|--------|--------|---------|----------|---------------|
| 0 | 0.1316 | 0.4172 | 0.7399 | 1.3098 | 2.2313 | 2.9886 | 3.1261 | 3.1416 |
| 0.01 | 0.417 | 0.4954 | 0.7579 | 1.3134 | 2.2326 | 2.9901 | 3.1277 | 3.1432 |
| 0.1 | 0.736 | 0.7542 | 0.8783 | 1.3437 | 2.2434 | 3.003 | 3.1416 | 3.1572 |
| 1 | 1.248 | 1.252 | 1.2871 | 1.5358 | 2.3265 | 3.1085 | 3.2567 | 3.2733 |
| 10 | 1.7228 | 1.7246 | 1.7406 | 1.8793 | 2.5388 | 3.4412 | 3.6423 | 3.6646 |
| 100 | 1.8568 | 1.8584 | 1.8721 | 1.994 | 2.6262 | 3.6134 | 3.8615 | 3.8892 |
| 1000 | 1.8733 | 1.8748 | 1.8883 | 2.0084 | 2.6376 | 3.6377 | 3.894 | 3.9227 |
| 10^{17} | 1.8751 | 1.8766 | 1.8901 | 2.01 | 2.6389 | 3.6505 | 3.8978 | 3.9266 |



(a)



(b)

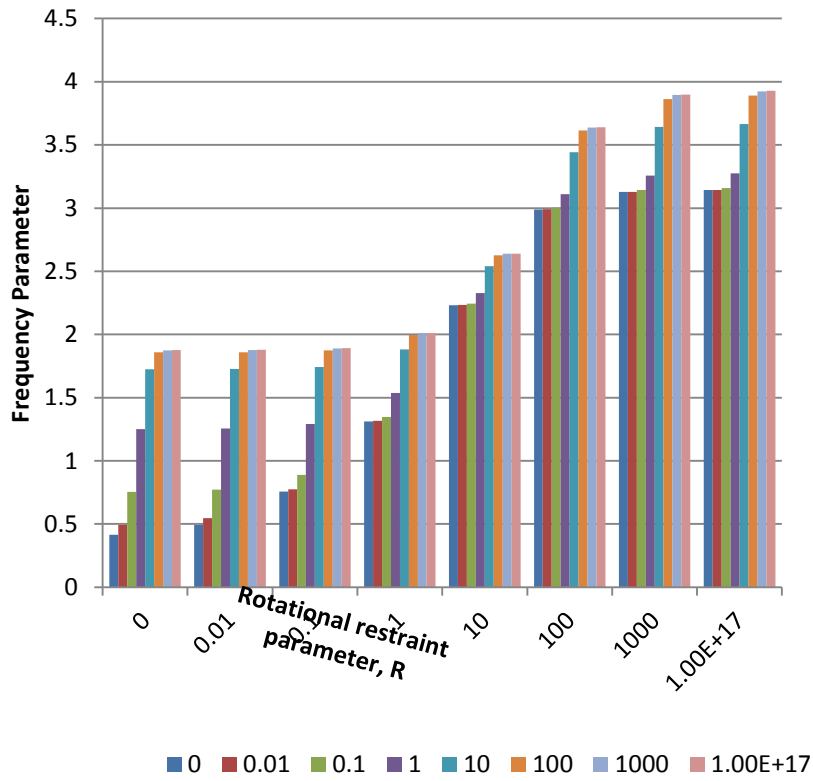


(c)

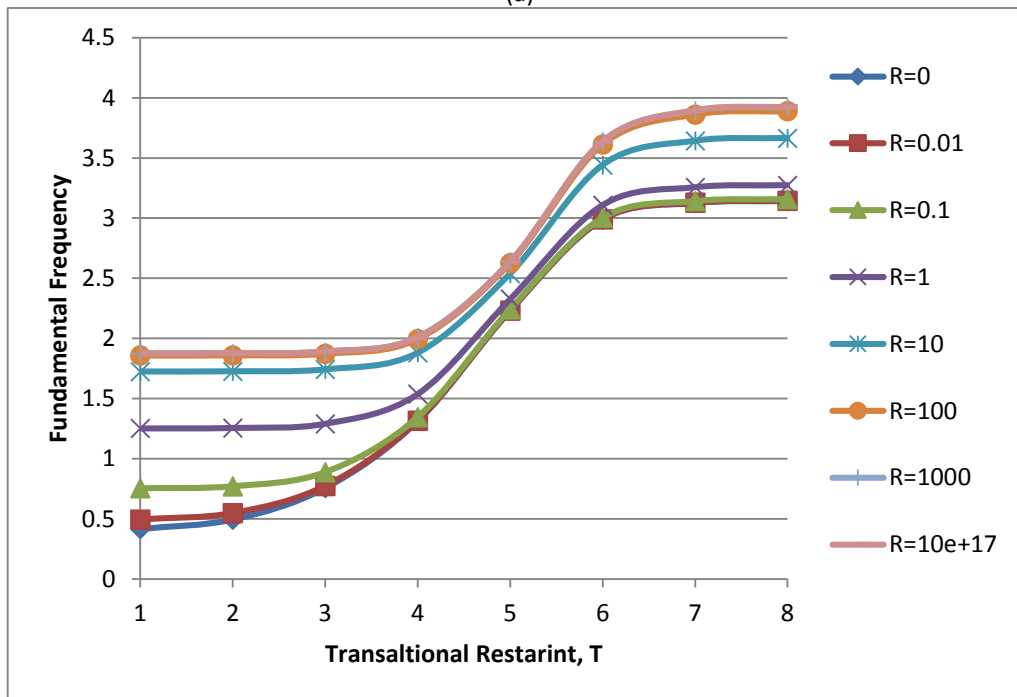
Fig. 3(a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter ($K=0.01$).

Table.3 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 0.1$

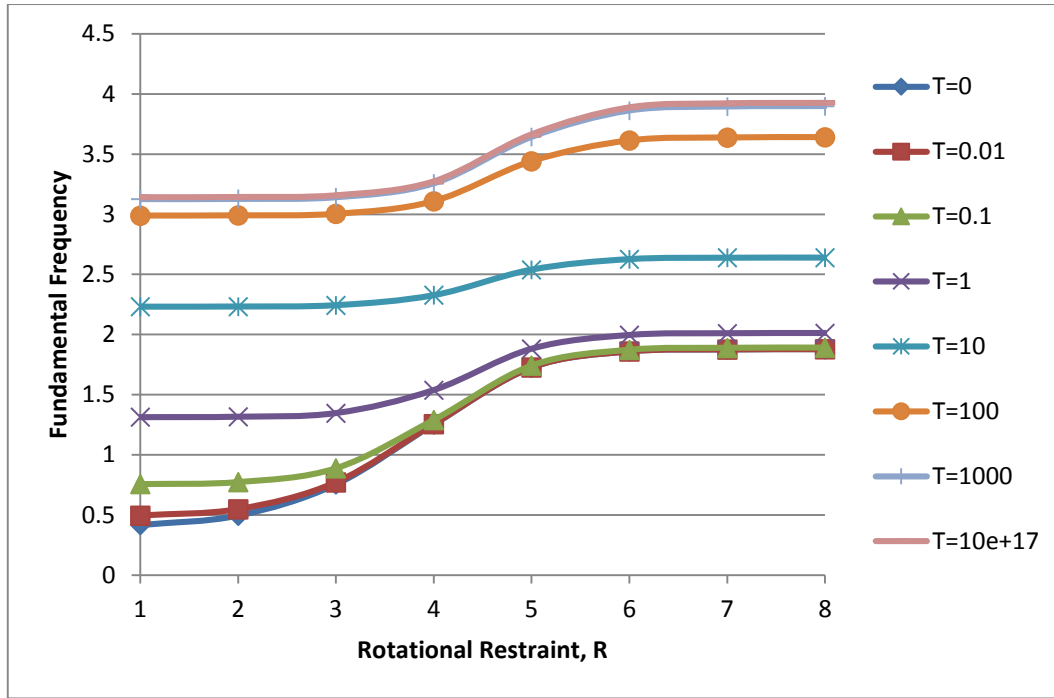
| R | $T = 0$ | $T = 0.01$ | $T = 0.1$ | $T = 1$ | $T = 10$ | $T = 100$ | $T = 1000$ | $T = 10^{17}$ |
|-----------|---------|------------|-----------|---------|----------|-----------|------------|---------------|
| 0 | 0.4162 | 0.4949 | 0.7575 | 1.313 | 2.2317 | 2.9893 | 3.1269 | 3.1424 |
| 0.01 | 0.4948 | 0.5477 | 0.7743 | 1.3166 | 2.2331 | 2.9907 | 3.1284 | 3.144 |
| 0.1 | 0.7541 | 0.7711 | 0.8891 | 1.3467 | 2.2439 | 3.0036 | 3.1423 | 3.158 |
| 1 | 1.252 | 1.2561 | 1.2908 | 1.5379 | 2.327 | 3.109 | 3.2573 | 3.274 |
| 10 | 1.7247 | 1.7265 | 1.7425 | 1.8808 | 2.5393 | 3.4416 | 3.6428 | 3.6652 |
| 100 | 1.8586 | 1.8601 | 1.8738 | 1.9953 | 2.6266 | 3.6136 | 3.8619 | 3.8897 |
| 1000 | 1.875 | 1.8765 | 1.8899 | 2.0097 | 2.6381 | 3.638 | 3.8945 | 3.9232 |
| 10^{17} | 1.8769 | 1.8784 | 1.8918 | 2.0114 | 2.6394 | 3.6408 | 3.8983 | 3.9271 |



(a)



(b)

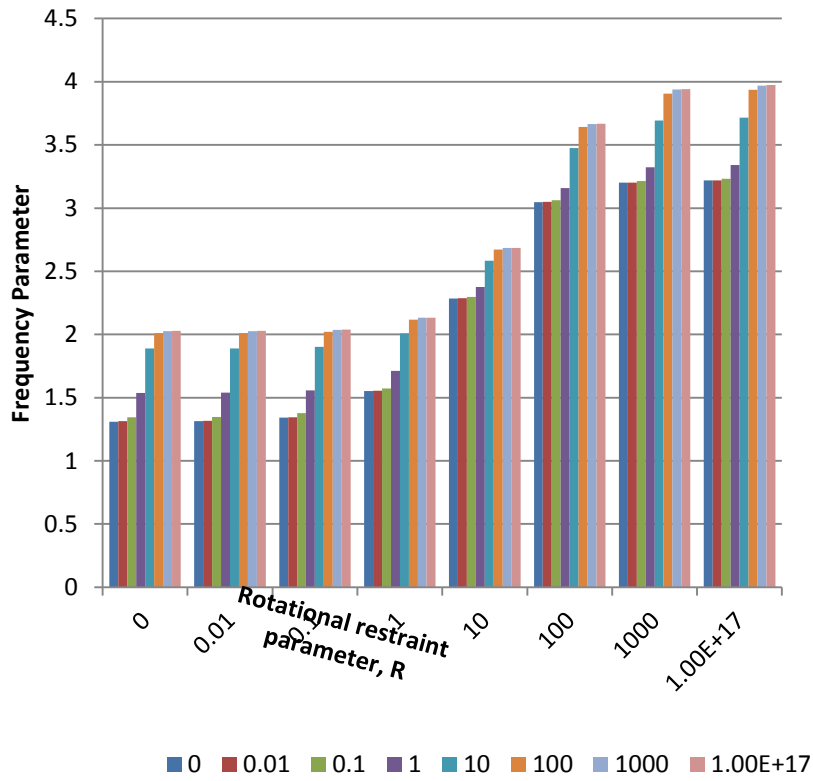


(c)

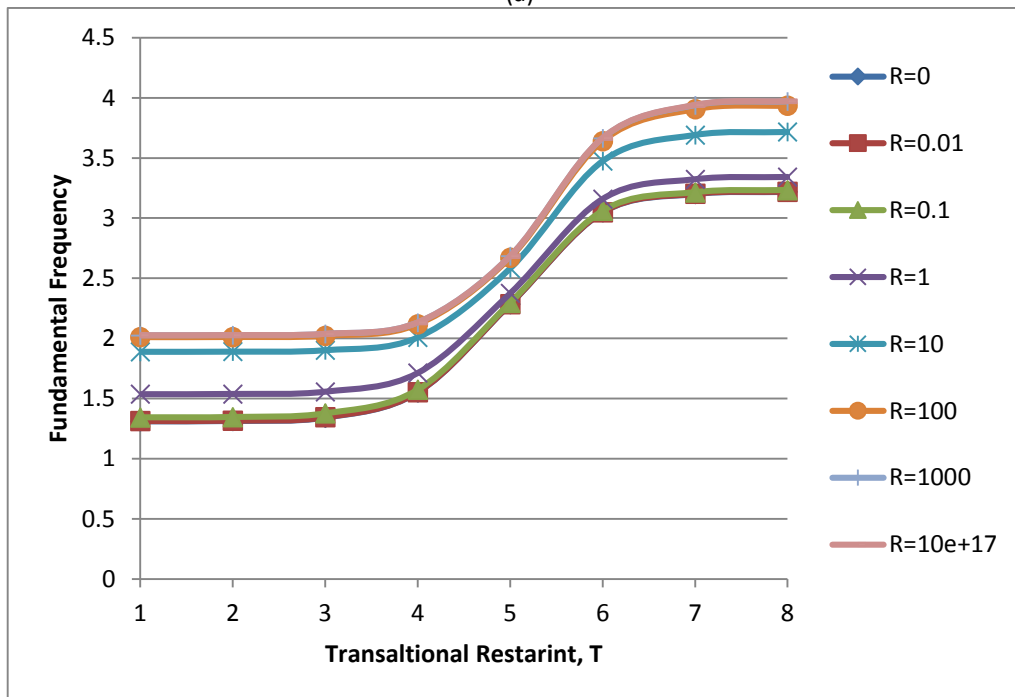
Fig. 4(a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter ($K=0.1$).

Table.4 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 1$

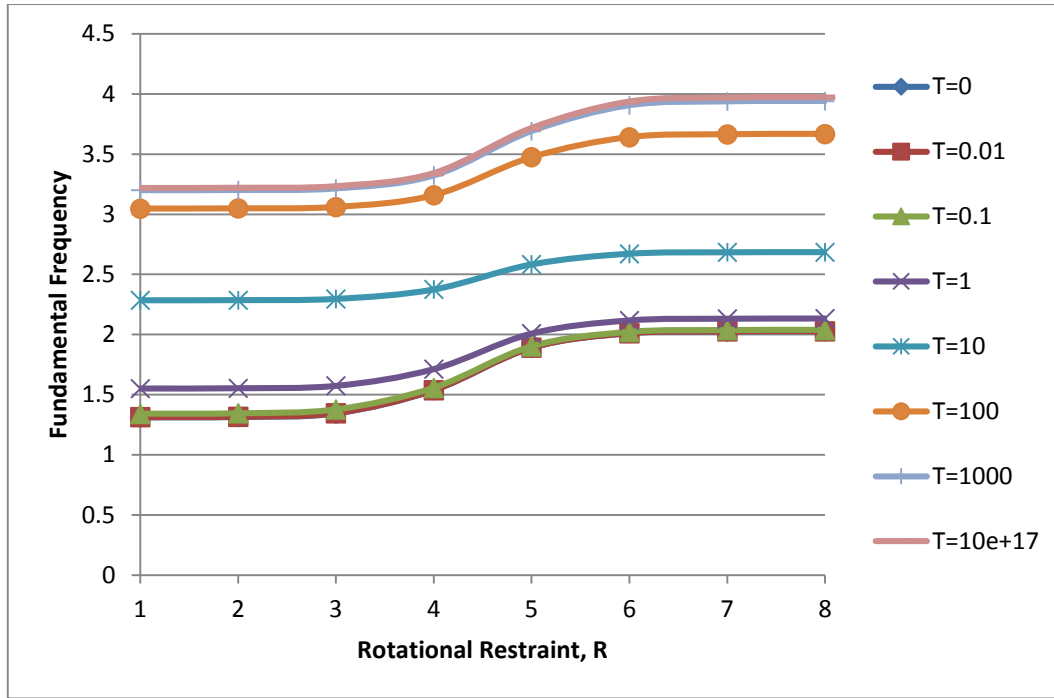
| R | $T = 0$ | $T = 0.01$ | $T = 0.1$ | $T = 1$ | $T = 10$ | $T = 100$ | $T = 1000$ | $T = 10^{17}$ |
|-----------|---------|------------|-----------|---------|----------|-----------|------------|---------------|
| 0 | 1.3104 | 1.3136 | 1.3414 | 1.5516 | 2.2852 | 3.0479 | 3.2008 | 3.2183 |
| 0.01 | 1.3139 | 1.3171 | 1.3447 | 1.5538 | 2.2863 | 3.0493 | 3.2023 | 3.2198 |
| 0.1 | 1.344 | 1.347 | 1.377 | 1.5736 | 2.2964 | 3.0612 | 3.2152 | 3.2328 |
| 1 | 1.5364 | 1.5385 | 1.5572 | 1.7125 | 2.3746 | 3.1591 | 3.323 | 3.3416 |
| 10 | 1.8891 | 1.8904 | 1.9022 | 2.0084 | 2.5828 | 3.4746 | 3.6919 | 3.7165 |
| 100 | 2.0102 | 2.0114 | 2.0219 | 2.118 | 2.6722 | 3.6413 | 3.9062 | 3.9363 |
| 1000 | 2.0256 | 2.0268 | 2.0371 | 2.1321 | 2.6841 | 3.6651 | 3.9383 | 3.9694 |
| 10^{17} | 2.0274 | 2.0285 | 2.0389 | 2.1337 | 2.6854 | 3.6679 | 3.942 | 3.9733 |



(a)



(b)

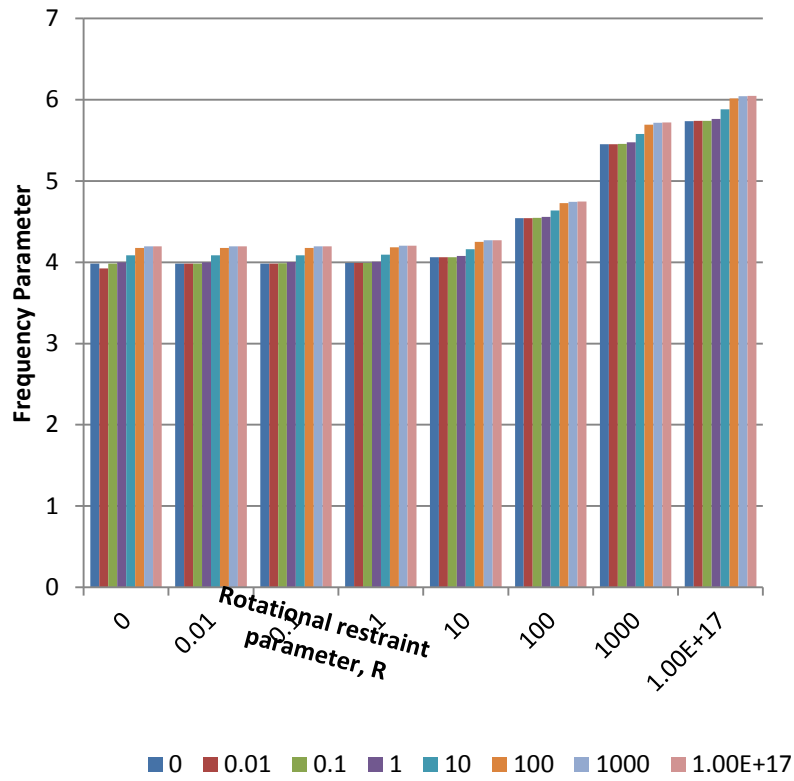


(c)

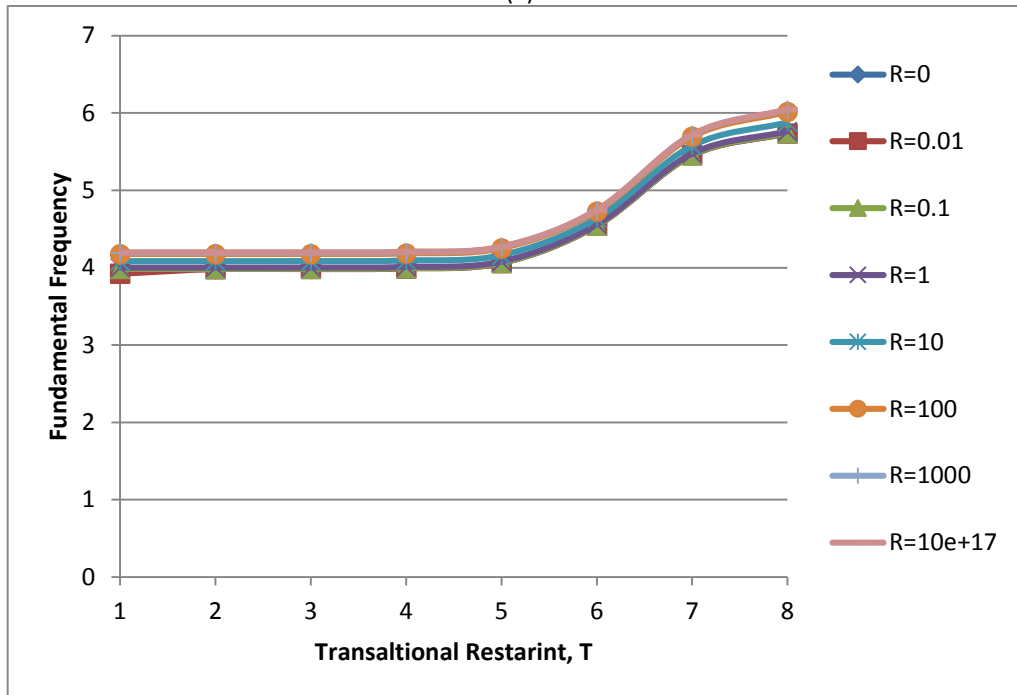
Fig. 5(a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter (K=1).

Table. 5 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 10$

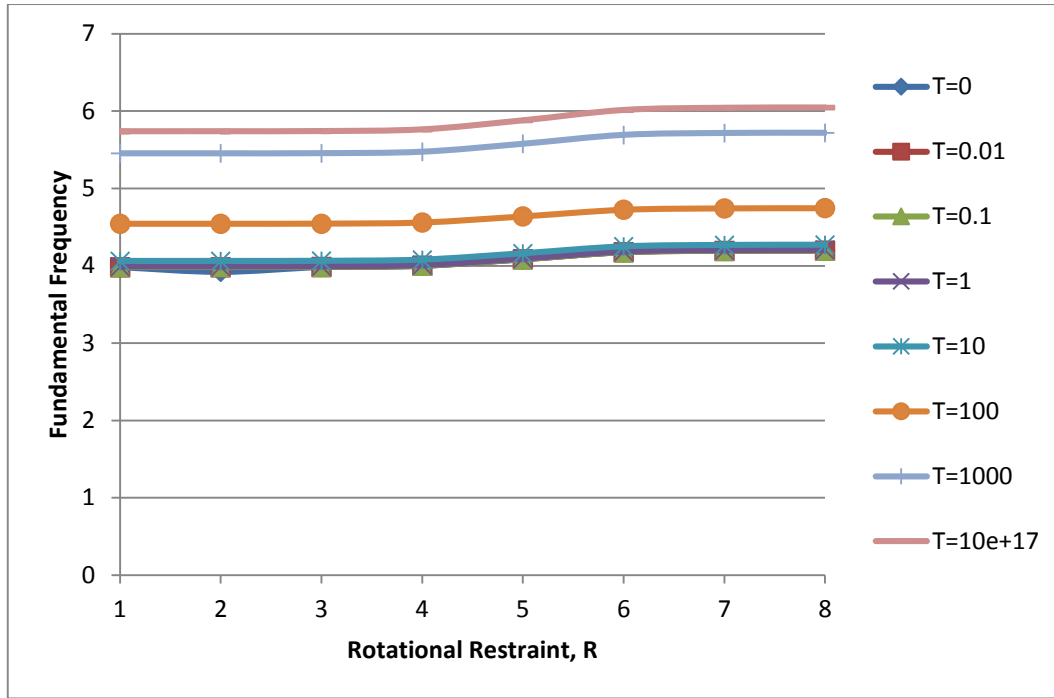
| R | T = 0 | T = 0.01 | T = 0.1 | T = 1 | T = 10 | T = 100 | T = 1000 | T = 10^{17} |
|-----------|--------|----------|---------|--------|--------|---------|----------|---------------|
| 0 | 3.9827 | 3.9828 | 3.9836 | 3.9909 | 4.0613 | 4.5431 | 5.454 | 5.7384 |
| 0.01 | 3.9229 | 3.983 | 3.9837 | 3.9911 | 4.0615 | 4.5433 | 5.454 | 5.7387 |
| 0.1 | 3.9846 | 3.9847 | 3.9855 | 3.9928 | 4.0632 | 4.5449 | 5.4562 | 5.741 |
| 1 | 4.0004 | 4.0005 | 4.0013 | 4.0086 | 4.0787 | 4.5597 | 5.4751 | 5.7626 |
| 10 | 4.0842 | 4.0843 | 4.085 | 4.0922 | 4.161 | 4.6389 | 5.5778 | 5.8806 |
| 100 | 4.1752 | 4.1753 | 4.176 | 4.1831 | 4.2505 | 4.7258 | 5.6934 | 6.0158 |
| 1000 | 4.1942 | 4.1942 | 4.1949 | 4.202 | 4.2692 | 4.7439 | 5.7178 | 6.0447 |
| 10^{17} | 4.1965 | 4.1966 | 4.1973 | 4.2043 | 4.2715 | 4.7462 | 5.7208 | 6.0483 |



(a)



(b)

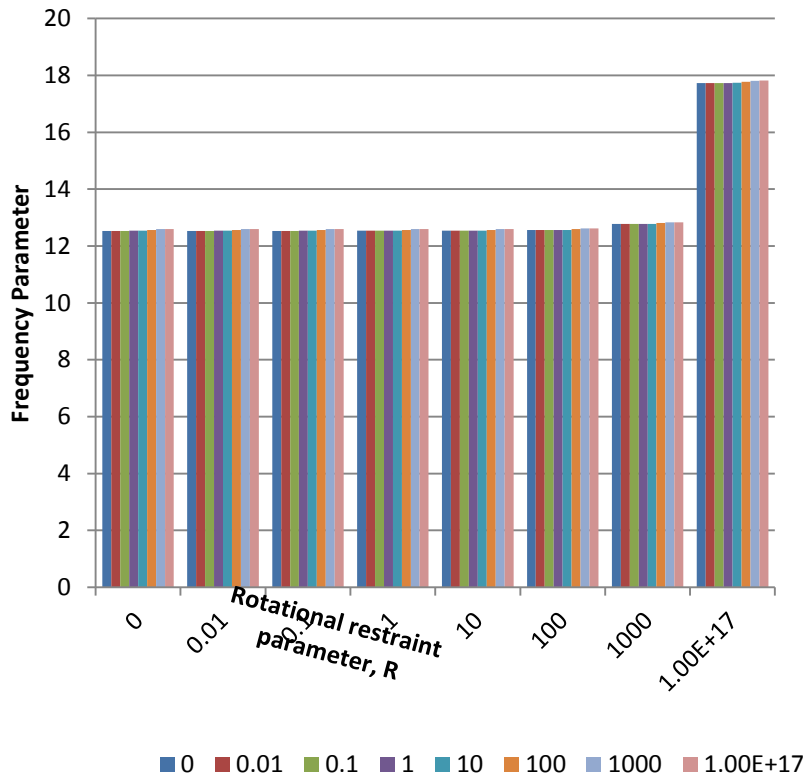


(c)

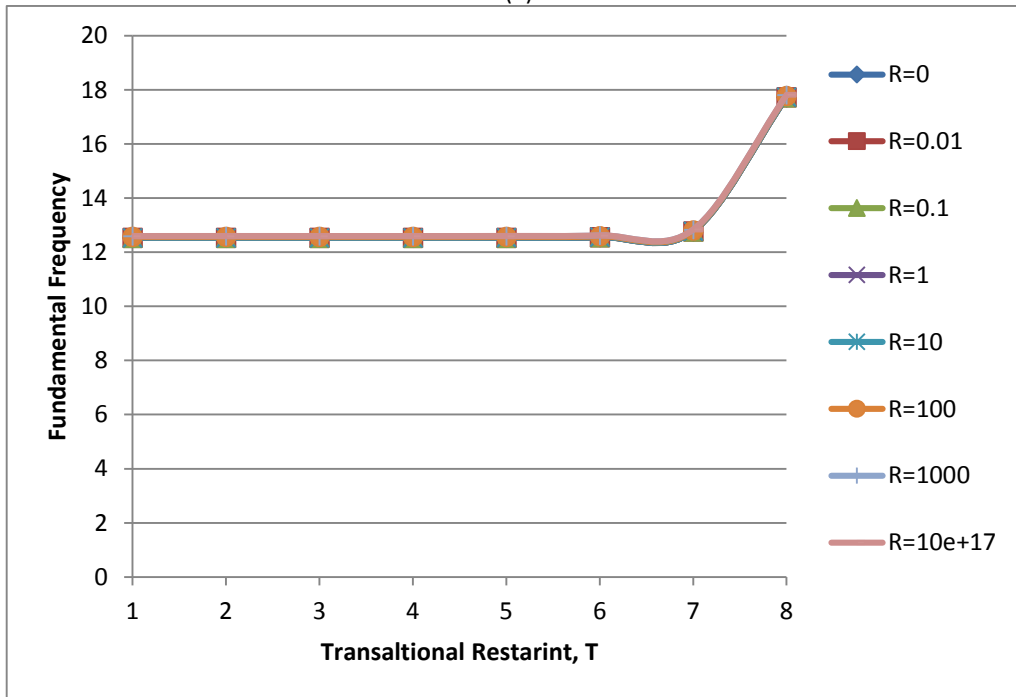
Fig. 6(a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter ($K=10$).

Table. 6 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 100$

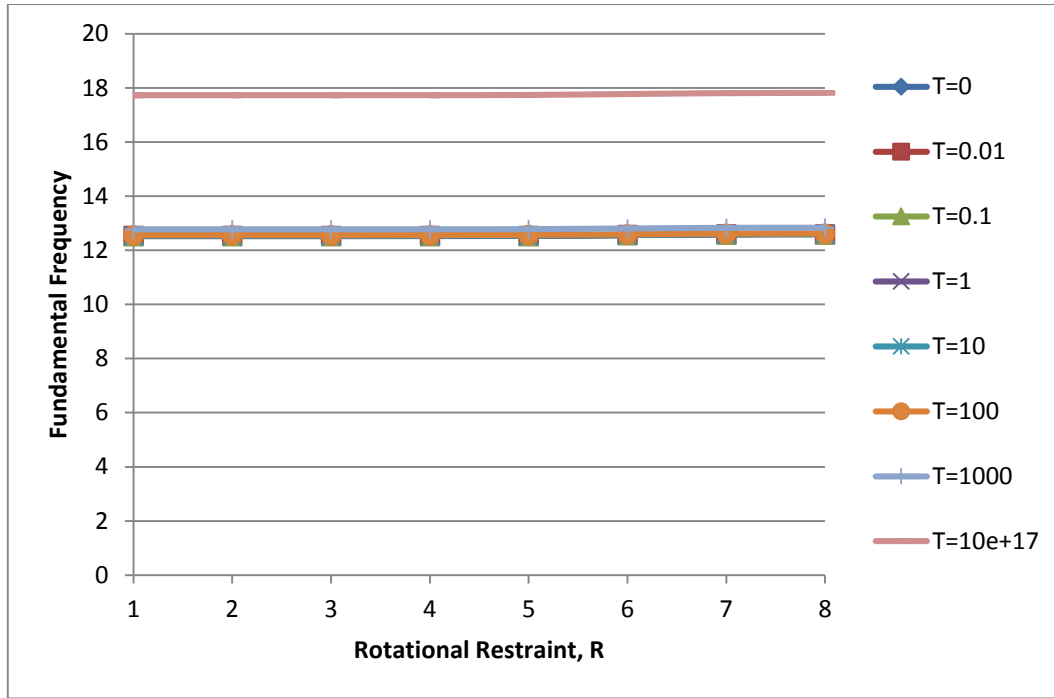
| R | $T = 0$ | $T = 0.01$ | $T = 0.1$ | $T = 1$ | $T = 10$ | $T = 100$ | $T = 1000$ | $T = 10^{17}$ |
|-----------|---------|------------|-----------|---------|----------|-----------|------------|---------------|
| 0 | 12.5339 | 12.5339 | 12.5339 | 12.5342 | 12.5364 | 12.5592 | 12.7758 | 17.7289 |
| 0.01 | 12.5339 | 12.5339 | 12.5339 | 12.5342 | 12.5364 | 12.5592 | 12.7758 | 17.7289 |
| 0.1 | 12.534 | 12.534 | 12.534 | 12.5342 | 12.5365 | 12.5592 | 12.7758 | 17.729 |
| 1 | 12.5345 | 12.5345 | 12.5345 | 12.5348 | 12.5371 | 12.5598 | 12.7764 | 17.7298 |
| 10 | 12.5396 | 12.5396 | 12.5396 | 12.5399 | 12.5421 | 12.5649 | 12.7814 | 17.737 |
| 100 | 12.5653 | 12.5654 | 12.5654 | 12.5656 | 12.5679 | 12.5906 | 12.8067 | 17.7734 |
| 1000 | 12.5913 | 12.5913 | 12.5913 | 12.5915 | 12.5938 | 12.6164 | 12.8332 | 17.8101 |
| 10^{17} | 12.597 | 12.5976 | 12.5971 | 12.5973 | 12.5996 | 12.6222 | 12.8378 | 17.8182 |



(a)



(b)

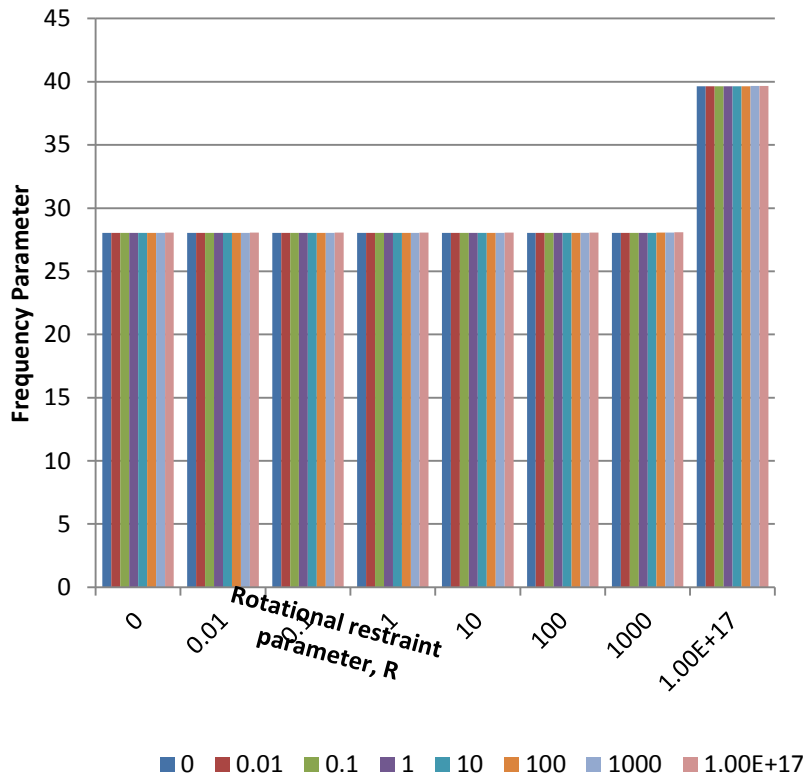


(c)

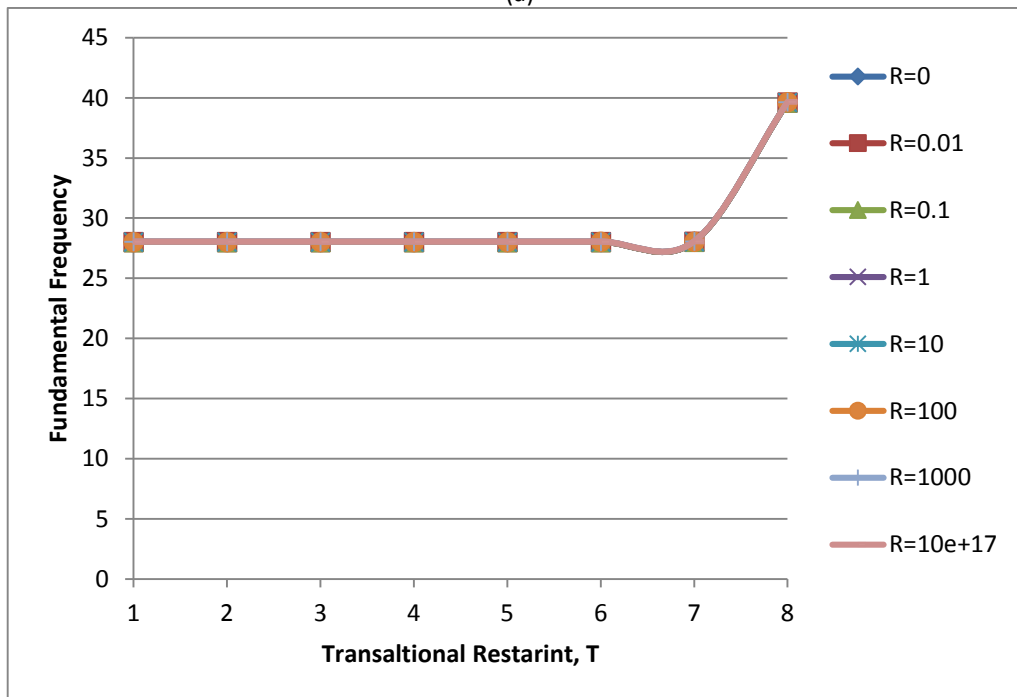
Fig. 7(a),(b) and (c). Variation of frequency parameter with rotational and translational restraints (R&T=0 to 10^{17}) for a given Warping parameter ($K=100$).

Table.7 First mode natural frequencies for various values of rotational and translational restraint parameters and for warping parameter $K = 500$

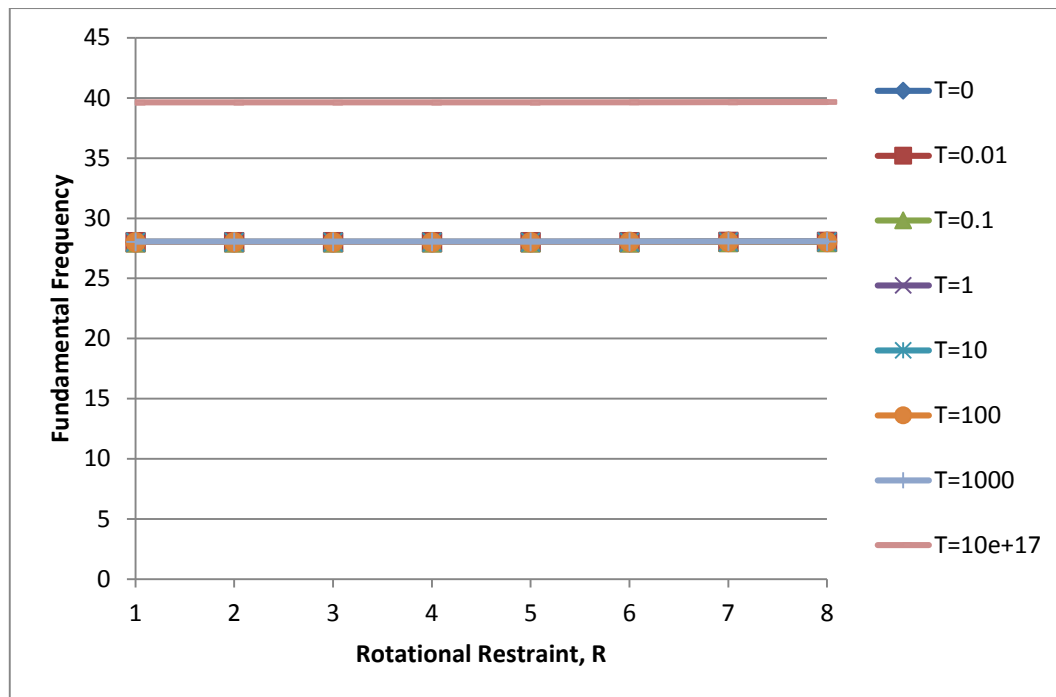
| R | $T = 0$ | $T = 0.01$ | $T = 0.1$ | $T = 1$ | $T = 10$ | $T = 100$ | $T = 1000$ | $T = 10^{17}$ |
|-----------|---------|------------|-----------|---------|----------|-----------|------------|---------------|
| 0 | 28.025 | 28.025 | 28.025 | 28.025 | 28.0253 | 28.0273 | 28.0477 | 39.6337 |
| 0.01 | 28.025 | 28.025 | 28.025 | 28.025 | 28.0253 | 28.0273 | 28.0477 | 39.6337 |
| 0.1 | 28.025 | 28.025 | 28.025 | 28.0251 | 28.0253 | 28.0273 | 28.0477 | 39.6337 |
| 1 | 28.0251 | 28.0251 | 28.0251 | 28.0251 | 28.0253 | 28.0274 | 28.0478 | 39.6337 |
| 10 | 28.0256 | 28.0256 | 28.0256 | 28.0256 | 28.0258 | 28.0278 | 28.0482 | 39.6344 |
| 100 | 28.0297 | 28.0297 | 28.0297 | 28.0287 | 28.0299 | 28.032 | 28.0524 | 39.6403 |
| 1000 | 28.0437 | 28.0437 | 28.0437 | 28.0437 | 28.044 | 28.046 | 28.0664 | 39.6601 |
| 10^{17} | 28.0531 | 28.0531 | 28.0531 | 28.0531 | 28.0533 | 28.0554 | 28.0757 | 39.6734 |



(a)



(b)



(c)

Fig. 8(a), (b) and (c). Variation of frequency parameter with rotational and translational restraints (R & $T=0$ to 10^{17}) for a given Warping parameter ($K=500$).

V. CONCLUSIONS

Spectral dynamic analysis of free torsional vibration of doubly-symmetric thin-walled beams of open section is carried out and detailed results of this study are presented in this paper suitable for use in design and also for checking approximate solutions obtained for their accuracy. For the case of a cantilever thin-walled beam of doubly-symmetric open cross-section undergoing free torsional vibrations and subjected to partial warping restraint, the spectral frequency equation is derived in this paper. The resulting transcendental frequency equation for the case of rotationally restrained cantilever with transverse restraint on the other end is solved for thin-walled beams of open cross section for varying values of warping parameter and the partial rotational and transverse restraint parameters. The influence of partial rotational restraint parameter R , transverse restraint parameter T and the warping parameter K on the free torsional vibration frequencies is investigated in detail and significant amount of numerical frequency data is generated. Using a MATLAB computer program developed to solve the spectral frequency equation derived in this study, numerical results for the first three modes of torsional natural frequencies for various values of R , T and warping K are obtained and are presented in both tabular as well as graphical form showing their parametric influence clearly. In comparison with the partial restraint parameters R and T , the warping parameter K is found to have significant effect on the torsional natural frequencies not only of the fundamental mode but also on higher modes as well.

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