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Development Of a Tube-Based Model Predictive Control for Linear Parameter Varying Systems Using a Lyapunov-Based Quadratic Terminal Cost Function

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ABSTRACT: *Model predictive control (MPC) technique for linear parameter-varying (LPV) systems has majorly been used to mitigate the computational complexity associated with nonlinear model predictive control (NMPC) method for nonlinear systems. However, the MPC technique for LPV systems faces challenges due to uncertainties in the future behavior of the LPV scheduling variables. Recent tube-based MPC approaches have sought to minimize this uncertainty but tend to be conservative due to the choice of terminal cost used, which might not adequately account for the variation of systems' parameters and thus, may hinder the achievement of control goals. To address this conservatism, this paper presents a tube-based model predictive control (TMPC) strategy for nonlinear systems within an LPV framework, utilizing a Lyapunov-based quadratic terminal cost function. In this study, a Lyapunov function that varies with system's scheduling parameters is used as the quadratic terminal cost in conjunction with a centre-based stage cost for an enhanced feasibility. The gain matrices required for the state-feedback controller needed for designing the terminal sets and the Lyapunov matrices required for the estimation of the terminal costs were derived through a linear matrix inequalities (LMI) optimization technique while the terminal sets were designed to be contractive to improve the robustness of the terminal controller. The developed TMPC algorithm is tested using simulations of an electrically-driven inverted pendulum and a two-tank system. This method is found to have successfully guided the states of the systems to desired trajectories while fulfilling the control objectives.*

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I. INTRODUCTION

Model predictive control (MPC) for linear parameter-varying (LPV) systems method has consistently been used to mitigate the complexities associated with nonlinear model predictive control (NMPC) for nonlinear system models (Jungers et al., 2011; Casavola et al., 2012; Su et al., 2012; Iles et al., 2015; Calderon et al., 2019; Morato et al., 2019a, 2091b). The MPC for LPV framework is also being used to address the discrepancies often suffered by real-world dynamic models between the actual system and its mathematical representation, as well as to correct uncertainties in the process, which can potentially compromise the closed-loop control system performance, leading to violations of operational constraints. However, the MPC for LPV embedding introduces its own challenges, particularly concerning the unpredictable future variation of the LPV scheduling variables throughout the prediction horizon (Morato et al., 2021).

To tackle this issue, a tube-based MPC design approach was proposed as used in the works of (Su et al., 2012; Brunner et al., 2013; Hanema et al., 2016, 2017, 2021; Abbas et al., 2021; Szederkenyi et al., 2022; Zhong et al., 2023; Lee and Jeong, 2024; Gao et al., 2024); this utilizes potential future state trajectories to minimize uncertainty in the scheduling variable, thereby ensuring stability and recursive feasibility. These attributes – stability and recursive feasibility – are crucial for the effective operation of control systems, as they provide the reliability and robustness needed for systems to function as intended amid varying conditions and disturbances. By upholding these characteristics, control systems can attain optimal performance, safety, and efficiency in managing complex dynamic systems. This tube-based MPC was however, despite its robustness to uncertainties and disturbances, discovered to have been overly cautious due to the choice of terminal costs used.

The choice of terminal cost adopted is a major concern in the tube-based MPC approach. In the works of (Hanema et al., 2016, 2017, 2021; Abbas et al., 2021), the controllers were discovered to be conservative due to the use of terminal costs constructed with parameter-independent weighted matrices that do not adequately capture the systems' dynamics. Meanwhile, poly-quadratic terminal cost function was adopted by (Nezami et al., 2022; Lee and Jeong, 2024) in their works, which is computationally complex. (Zhong et al., 2023; Gao et al., 2024) used a data-driven or learning based method in developing their terminal costs, however, learned models may not be able to adequately capture the system dynamics, coupled with a potential struggle to handle disturbances.

According to (Wada et al., 2004; Danielson, 2021), terminal cost behaves like a Lyapunov function for model predictive control by playing a critical function in shaping the behavior of the MPC controller which is essential for achieving desired control objectives such as stability, performance, constraint satisfaction and robustness. Therefore, selection of terminal cost function is very crucial and is worth focusing on, in model predictive control. A review according to Lazar and Tetteroo (2018), revealed the common methodology for determining terminal sets and costs in NMPC, noting that past works often linearize the nonlinear system around a zero equilibrium and specify the terminal cost with a linear state feedback controller, often realized as a linear quadratic regulator (LQR) for stability. Typically, a parameter-independent weighted matrix is often utilized instead of a parameter-dependent Lyapunov matrix in the determination of terminal costs, which can misrepresent the system's dynamics and potentially compromise stability (Bao et al., 2022).

This study is aimed at investigating and proposing a tube-based model predictive control (TMPC) strategy which offers a Lyapunov-based quadratic terminal cost function, designed with contractive terminal sets – for an improved robustness of the controller. This approach intends to be less conservative and easy to compute, without complexity to the resulting optimization problem and can as well, captures the dynamics of the system adequately while concentrating on the use of system models, since most LPV controller designs and their deployment in the real-world have been originally developed within a state-space framework. The focus is restricted to discrete-time state-space LPV models for easy design, improved accuracy, analysis, flexibility, robustness, reliability, scalability, improved safety and lower cost.

The rest of the paper covers the methods and materials, results and discussions, and conclusion.

II. MATERIAL AND METHODS

This section introduces the preliminaries, the MPC approach, initialization approach and numerical examples.

2.1 Preliminaries

This subsection consists of notations, concept of LPV framework and terminal controller design.

Notation

The under listed are the notations used in this paper and their meanings.

i. $\mathbb{R}$ represents the real number set, R_+ denotes the set of non-negative real numbers, and $\mathbb{N}$ signifies the non-negative integers set, including zero.
ii. A PC-set is defined as a convex and compact set whose interior contains the origin and is non-empty while a polyhedron is a convex set formed by the intersection of multiple half-spaces, with a polytope, S being a compact polyhedron.
iii. Sequences are represented in a compact form as $\{X_i\}_{i=a}^{b} = \{X_a, X_{a+1}, \dots, X_b\}$. X_a and X_b are the initial cross-section and final cross-section respectively.
iv. θ , $\mathbb{N}$, $\mathbb{U}, x$ and $u$represent the scheduling variable, state constraint, input constraint, state signal and input signal respectively.
v. θ, x and u are related in a way that a map $\mu\colon \mathbb{X} \times \mathbb{U} \longrightarrow \Theta$ exists such that $\theta = \mu(x, u)$.
vi. i and N denote the prediction horizon and number of prediction horizons respectively, while γ and λ are the contractivity factors evaluated at the final step of the prediction horizon and time horizon respectively.

Linear Parameter Varying Framework

A constrained system represented with the state-space equation of the form:

$$x(k+1) = f\big(x(k), u(k)\big), x(0)x_0 \tag{1}$$

could be approximated using a linear parameter-varying (LPV) state-space that is modeled in the form of the equation:

$$x(k+1) = A\big(\theta(k)\big)x(k) + B\big(\theta(k)\big)u(k) \tag{2}$$

with $u(k)$ being the control input, $x(k)$ being the state variable, $\theta(k)$ being the scheduling parameter and $A(\theta) = A_0 + \sum_{i=1}^{n\theta} A_i$, such that $\{A_i\}_{i=1}^{n\theta}$ are real matrices with dimensions that are compatible.

Terminal Controller Design

A system subject to constraints, with dynamics given by the state-space equation:

$$x(k+1) = A(\theta(k))x(k) + BK_l(x(k), \theta(k), k), \theta(k) = \mu(x(k)) \quad (3)$$

has a controller K_l that ensures regional and asymptotical stability for the closed-loop system's origin or equilibrium point which is designed in consideration of a control problem. The idea is that, if the LPV framework is stabilized by the MPC controller, then the nonlinear system embedded is stabilized in conformity with principles of embedding.

The terminal controller is computed in accordance with the stability condition for discrete-time LPV systems as described in (Chesi et al., 2005; Chesi, 2010; Bao et al., 2022). The condition involves finding solution to a linear matrix inequality (LMI) problem to find matrices $Q_i > 0, X_i \in \mathbb{R}^{n_x \times n_x}, L_i \in \mathbb{R}^{n_u \times n_x}, Y_i \in \mathbb{R}^{n_u \times n_x}, Z_i \in \mathbb{R}^{n_u \times n_x}, i = 1, \ldots, q$ such that

$$\begin{bmatrix} G & * & * & * & * \\ A_i^T X_i & R & * & * & * \\ -L_i & H & Z_j + Z_j^T & * & * \\ Q^{\frac{1}{2}} X_i & 0 & 0 & I & * \\ R^{\frac{1}{2}} L_i & 0 & 0 & 0 & I \end{bmatrix} > 0 \quad (4)$$

for $\forall i, j = 1, \ldots q$, * indicates some symmetric blocks which are left out for conciseness, $G = X_i + X_i^T - Q_i$, $H = Z_j^T B^T - Y_j$ and $R = Q_j - B_i Y_j + (B_i Y_j)^T$. The terminal controller gain is computed as $K_i = L_i X_i^{-1}$ while the parameter-dependent Lyapunov matrix is given as $P_i = Q_i^{-1}$.

2.2 The MPC Approach

The MPC methodology for LPV systems is outlined under this subsection.

Tubes Construction

To formulate a model predictive control (MPC) approach for LPV framework, we need to create a set of feasible tubes that adapt to time-varying state constraints. The tubes represent sets of possible states or trajectories that the system can occupy which ensures or guarantees stability and robustness of the control system. These tubes can be defined as:

$$\mathcal{T}_N = (x|\mathbb{X}) = \left\{ T \middle| \begin{array}{l} with\ X_0 = \{x\}, X_N \subseteq X_f\ and \\ \forall i \in [0 \ldots N-1]: \theta_i = \mu(\mathbb{X}_i) \end{array} \right\} \quad (5)$$

with X_f being the terminal set.

A particular tube would be chosen from the set as defined in (5) to optimize a specific performance metric for control. This is achieved by solving a tube-related problem in the form of:

$$V(x|\mathbb{X}) = \min_T J_N(T, \mathbb{X}) \text{ subject to: } T \epsilon \mathcal{T}(x|\mathbb{X}) \quad (6)$$

The cost function, $J_N(x, \mathbb{X})$ is defined as:

$$J_N(x, \mathbb{X}) = \sum_{i=0}^{N} \ell(X_i, K_i, \mu(\mathbb{X}_i)) + F(X_N) \quad (7)$$

with ℓ being the stage cost and F being the terminal cost.

Stage Cost Design

Stage cost is always used in specifying a performance target and for the purpose of this work, a centre-based stage cost is considered owing to its suitability for quadratic programming. The centre-based stage cost for a set of specified dimension, $X = z + \alpha S\ with\ S = convh\{s^{-1}, \ldots s^{-q_s}\}$, can be constructed as:

$$\ell_p^{cb}(X, K, \Theta) = \| Qz \|_p^{r(p)} + P|\alpha|^{r(p)} + \frac{1}{q_\theta q_s} \sum_{i=1}^{q_s} \sum_{j=1}^{q_\theta} RK(x^i, \theta^j) \|_p^{r(p)} \quad (8)$$

such that $p \in \{2, \infty\}$, matrices (Q, R) and scalar P – which is greater than zero – are tuning parameters with full column rank.

Terminal Set and Cost Design

In MPC, a terminal set, also known as a terminal region or target set, refers to a specific region in the state space that the system is steered towards, and remains within, once reached. The terminal set ensures stability and feasibility of the MPC problem by providing a target region for the system to converge to, most especially for constrained systems.

A terminal set constraint $X_N \subseteq X_f$ is part of (5) to ensure recursive feasibility of (6). The terminal set is computed via two algorithms namely: Robustly controlled positively invariant set computation and determination of robustly controlled invariant sets over N steps. The robustly controlled positively invariant set computation is adequately described in the work of (Bao et al., 2022).

Algorithm 1 Determination of robustly controlled invariant sets over N steps

Required input: $\{A\}_{i=1}^{q}, \{B\}_{i=1}^{q}$, sets $\mathbb{X}, \mathbb{U}$, Ω_{max}, and contractivity (λ). The procedure for determining the contractivity (λ) of a terminal set is contained in an algorithm as proposed by Miani and Savorgnan (2005), and can be defined as $\lambda = inf\{\gamma \geq 0 | \mathrm{I}(X, \hat{\mu}(\breve{\mathbb{X}}) | K) \subseteq \gamma X\} < 1$ while Ω_{max} is determined by the algorithm proposed by Bao et al.(2022).

Expected output: Robustly controlled invariant set over N steps, C_N.

1. Make $i = 0$ and $C_0 = \Omega_{max}$ and ensure the matrices F_0, g_0 be the half-space characterization of C_0, that is, $C_0 = \{x \in \mathbb{R}^n : F_0 x \leq g_0\}$.
2. Calculate the expanded set $P_i \subset \mathbb{R}^{n_x+n_u}$:

$$P_i = \left\{ \begin{array}{l} (x,u) \in \mathbb{R}^{n_x+n_u}: \begin{bmatrix} F_i(A_1 x + B_1 u_1) \\ F_i(A_2 x + B_2 u_2) \\ \cdot \\ \cdot \\ \cdot \\ F_i(A_q x + B_q u_q) \end{bmatrix} x \\ \leq \lambda \left(\begin{bmatrix} g_i \\ g_i \\ \cdot \\ \cdot \\ \cdot \\ g_i \end{bmatrix} \right) \end{array} \right\}$$

3. Extract the projection $P_i^{(n)}$ of P_i on $\mathbb{R}^{n_x}$:
$$P_i^{(n)} = \{x \in \mathbb{R}^{n_x} : \exists u \in \mathbb{U} \; s.t. (x,u) \in P_i\}.$$
4. Make $C_{i+1} = P_i^{(n)} \cap \mathbb{X}$. Allow F_{i+1}, g_{i+1} to be the half-space characterization of C_{i+1}, that is, $C_{i+1} = \{x \in \mathbb{R}^{n_x} : F_{i+1} x \leq g_{i+1}\}$.
5. If $C_{i+1} = C_i$, then stop and make $C_N = C_i$. Otherwise, continue.
6. Stop, provided $i = N$. Otherwise, proceed.
7. Make $i = i + 1$. Then, proceed to step 2.

Note that the N-step robustly invariant set is an expanded form of the robustly controlled positively invariant set to increase the domain of attraction (DOA) of the computed terminal set such that the terminal controller would be able to discriminate against disturbances and ensure system stability.

The terminal cost on its own is defined as:

$$F(X) = \frac{\Psi_{X_f}(X)}{1-\lambda} \left(x(k)^T P(\theta(k)) x(k)\right) \tag{9}$$

where $\Psi_{X_f} = \inf\{\gamma \geq 0 | X \subseteq \gamma S\}$, $P(\theta(k))$ is the parameter-dependent Lyapunov matrix and $x(k)$ is the terminal set. As evident in (9), the terminal cost function is parameter-dependent, with the Lyapunov matrix, P varying according to the scheduling parameter while the coefficient $\frac{\Psi_{X_f}(X)}{1-\lambda}$, restricts the magnitude of the terminal cost coefficient (Raphael et al., 2015). This approach is intended to offer flexibility and efficiency in solving control and optimization problems and to enhance the feasibility and robustness of the terminal controller such that any given system's dynamics or trajectory can reach desired states through the contractiveness of the terminal sets. The contractivity factor, λ which quantifies the rate at which system state trajectories converge to the origin, has a value range between 0 and 1.

The Main MPC Algorithm

The MPC procedure is summarily captured by the main algorithm.

Algorithm 2 MPC Procedure

Required: $N \in [1 \ldots \infty)$

1. Estimate $\mathbb{X}(0)$ based on the initial state, $x(0)$; while if step 4 is a realizable approach or procedure at k = 0, which means that $\tau_N(x(0)|\mathbb{X}_0) \neq \{\}$.
2. $k \leftarrow 0$
3. **begin loop**

4. Solve (6) to acquire $T^{\#} \epsilon\ \tau_N(x(k)|\mathbb{X}(k))$
5. Apply $u(k) = K_0{}^{\#}(x(k), \theta(k) = u_0{}^{\#}$ to system (1)
6. $\nabla i \ \epsilon\ N_{[0,\ N-1]}$: $\quad \mathbb{X}_i(k+1) \leftarrow \begin{Bmatrix} \mathbb{X}_{i+1}(k), & i < N-1 \\ \breve{\mathbb{X}}, & i = N-1 \end{Bmatrix}$
7. $k \leftarrow k+1$
8. **stop loop**

2.3 Initialization Approach

The controller initialization approach considered for this study is the bounded rate of variation method for its smooth transition from the initial state to the desired trajectory, thereby reducing abrupt changes and its contribution of reduction of overshoot and oscillations.

Let $\delta \in \mathbb{R}_+^{n_x}$ indicates a bound on the rate of variation of the state variable. Then, with an initial system state $x(0) = x_0$ evaluated, the corresponding initial constraint sequence $\mathbb{X}(0)$, can be computed as:

$$\forall i \ \in \ \mathbb{N}_{[0,N]} : \mathbb{X}_i(0) = \begin{Bmatrix} \{x_0\} & i = 0, \\ (\mathbb{X}_{i-1}(0) \oplus \triangle(\delta)) \cap \mathbb{X}, & i > 0, \end{Bmatrix} \tag{10}$$

$$\nabla(\delta) = \{x \in \mathbb{R}^{n_x} | \forall i \in \mathbb{N}_{[1,n_x]} : |x_i| \ \leq \ \delta_i\}. \tag{11}$$

The bounded rate of variation (ROV) construction would be done in such a fashion that a suitable value for δ must be identified, one that strikes a balance between allowing sufficient state variation and keeping scheduling sets reasonably sized. It should be of great service to mention that if there exists, a sectioning of the state vector $x = (x_1, x_2)$ in such a way that $\mu(x) = \mu(x_1)$, in that sense, it is adequate to enforce bounded rate of variation (ROV) on the x_1-segment alone.

2.4 Numerical Examples

Within the context of this section, the proposed algorithms would be validated, along with the algorithms from Hanema et al. (2021) and Nezami et al. (2022) with two numerical examples literatures namely, an electrically-driven inverted pendulum and a two-tank system. A 2017 Intel Core i7 CPU at 3.2 GHz, with Yalmip integrated and MATLAB serving as the Quadratic programming (QP) and Linear programming (LP) solver were used in obtaining the results.

An Electrically-driven Inverted Pendulum

This example focuses on controlling the angle q, of the inverted pendulum system shown in Figure 1, which is driven by an electric motor. The state vector is defined as:

$$x = \begin{bmatrix} q \\ \dot{q} \\ i \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{12}$$

where the angle of the pendulum is q, while $\dot{q}$ is the angular velocity, and I is the motor current. The dynamics of the system could be established in a discretised LPV embedded pattern as:

$$x(k+1) = \left(\begin{bmatrix} 1.00 & 0.04 & 0.07 \\ 0 & 0.90 & 2.64 \\ 0 & -0.02 & 0.64 \end{bmatrix} + \begin{bmatrix} 0.08 & 0 & 0 \\ 3.81 & 0.08 & 0 \\ -0.05 & 0 & 0 \end{bmatrix} \theta(k) \right) x(k) + \begin{bmatrix} 0 \\ 0.69 \\ 0.32 \end{bmatrix} u(k),$$

$$\theta(k) = sinc(x_1(k)) \tag{13}$$

The system operates under the following constraints

$$\mathbb{X} = \{x \in \mathbb{R}^3 | |x_1| \leq 2\pi, |x_2| \leq 18, |x_3| \leq 6\},$$
$$\mathbb{U} = \{u \in \mathbb{R} | |u| \leq 8\}. \tag{14}$$

Simulation parameters for the electrically driven inverted pendulum were set as follows:

i. The prediction horizon was configured to N = 6 while the tuning parameter was adjusted to Q = $\begin{bmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, with p = 5.
ii. The computation of a contractive terminal set X_f, is illustrated in:

$$\breve{\mathbb{X}} = \left\{x \in \mathbb{R}^3 \middle| |x_1| \leq \tfrac{1}{2}\pi, |x_2| \leq 18, |x_3| \leq 2\right\}$$

X_f is a contractive set with 24 vertices, stabilized by the linear state feedback controller $u = Kx$.

iii. The system started at an initial state $x_0 = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}$, which is an unstable equilibrium point.

iv. With the bounded ROV technique, the initial state constraint sequence and the associated scheduling sets were generated, using $\delta = \left[\frac{1}{4}\pi \ \infty \ \infty\right]^T$.

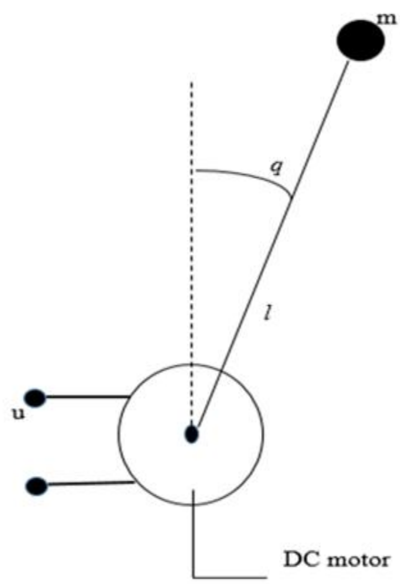

Figure 1: An electrically-driven inverted pendulum

A Two-tank system

In the two-tank system as shown in Fig. 2, the desire is to control the levels of liquid h_1 and h_2 at a prescribed set point while the input flow, u serves as the control input. A_1 and A_2 are the cross-sectional areas for pipe 1 and pipe 2 respectively, while S_1 and S_2 are the cross-sectional areas for tank 1 and tank 2 respectively. The discrete-time LPV embedding of the system yields:

$$x(k+1) = \left(\begin{bmatrix} 0.86 & 0 \\ 0.22 & 0.97 \end{bmatrix} + \begin{bmatrix} 0.12 & 0 \\ -0.19 & 0 \end{bmatrix}\theta_1(k) + \begin{bmatrix} 0 & 0 \\ 0 & 0.02 \end{bmatrix}\theta_2(k)\right)x(k) + \begin{bmatrix} 0.36 \\ 0 \end{bmatrix}u(k),$$
$$\theta(k) = \mu(x(k)) \quad (15)$$

where θ_1 and θ_2 are the scheduling variables corresponding to h_1 and h_2 respectively, while μ is the scheduling map. The intention is to take h_2 to a reference value of 115cm with the equilibrium state, x_{ss} being $\begin{bmatrix} 22.72 \\ 115 \end{bmatrix}$ and equilibrium input, u_{ss} of 1.90. Translated state and input variables: $\tilde{x} = x - x_{ss}$, $\tilde{u} = u - u_{ss}$ are introduced to ensure that controlling $\tilde{x}$ to zero is tantamount to controlling x to the desired set point.

The settings used for the simulation include:

i. The prediction horizon used was N = 10 while the tuning parameter Q = $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ was used and P = 5 was set for the centre-based stage cost.

ii. A controlled contractive terminal set X_f with a representation of 6 vertices was computed with $\breve{\mathbb{X}} = \mathbb{X}$ for the tube-based MPC

iii. The bounded rate of variation (ROV) approach with $\delta = [2 \ \ 25]^T$ was used for the initialization of the tube-based MPC.

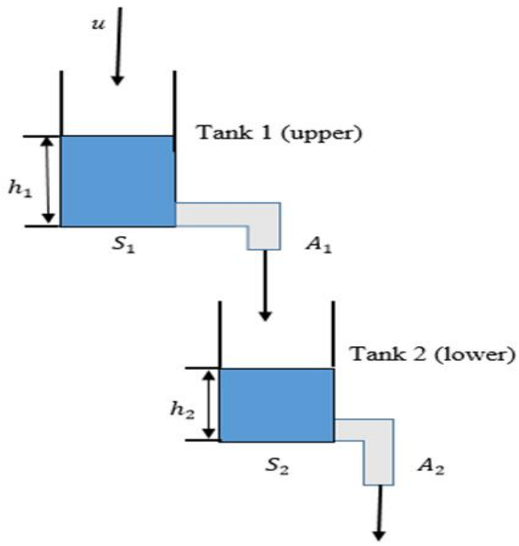

Figure 2: A two-tank system

III. RESULTS

This section reveals the results obtained for the electrically driven inverted pendulum and the two-tank system.

3.1 Results for the Electrically-driven Inverted Pendulum

From the results as shown in Figure 3, Figure 4, Figure 5, Table 1, Table 2 and Table 3, the trajectories of the three states of the electrically driven inverted pendulum – namely angle of pendulum, velocity of pendulum and motor current – as produced by the controller from this work, were able to effectively track the set points without any significant deviation or steady state error when compared with the trajectories produced by controllers from the algorithms of Hanema et al. (2021) and Nezami et al. (2022) – which exhibit higher amount of steady-state error.

For the response time, the controller from this work was able to drive the three states to within stable equilibrium points at the fastest time followed by the controller from Hanema et al. (2021), while the controller from Nezami et al. (2022) was the slowest to drive the states to the stable equilibrium points.

Meanwhile, all the three controllers were able to satisfy the specified constraints of the system

Table 1: Results for angle of pendulum of the electrically-driven inverted pendulum

Performance metrics	Controller from Hanema et al. (2021)	Controller from this work	Controller from Nezami et al. (2022)
Response time	0.52 seconds	0.48 seconds	0.92 seconds
Maximum Steady state error	0.0665 radians	0.0180 radians	0.1050 radians
Constraints satisfaction	Good	Good	Good
Maximum overshoot/undershoot	–––	–––	–––

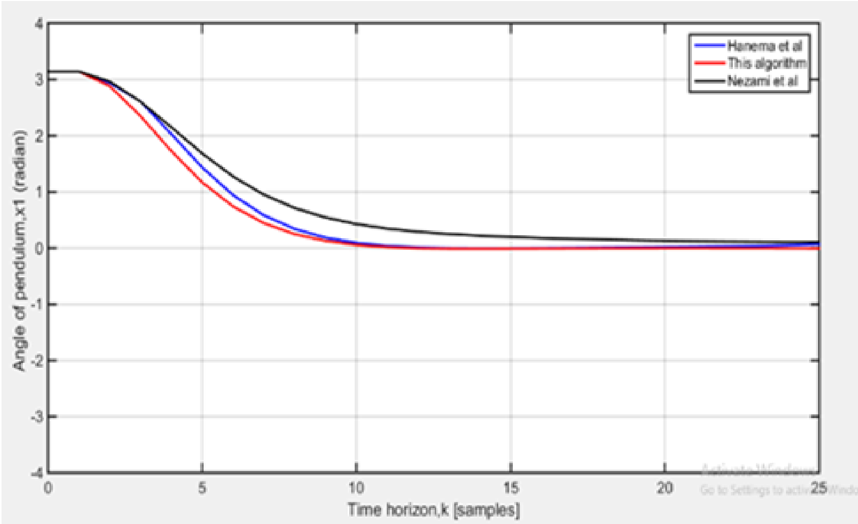

Figure 3: Trajectories for angle of pendulum of the electrically-driven inverted pendulum

Table 2: Results for velocity of pendulum of the electrically-driven inverted pendulum

Performance metrics	Controller from Hanema et al. (2021)	Controller from this work	Controller from Nezami et al. (2022)
Response time	0.56 seconds	0.52 seconds	0.92 seconds
Maximum steady state error	0.4710 rad/secs	0.0600 rad/s	0.0957 rad/s
Constraints satisfaction	Good	Good	Good
Maximum overshoot/undershoot	14.56 rad/secs	14.63 rad/secs	11.57 rad/secs

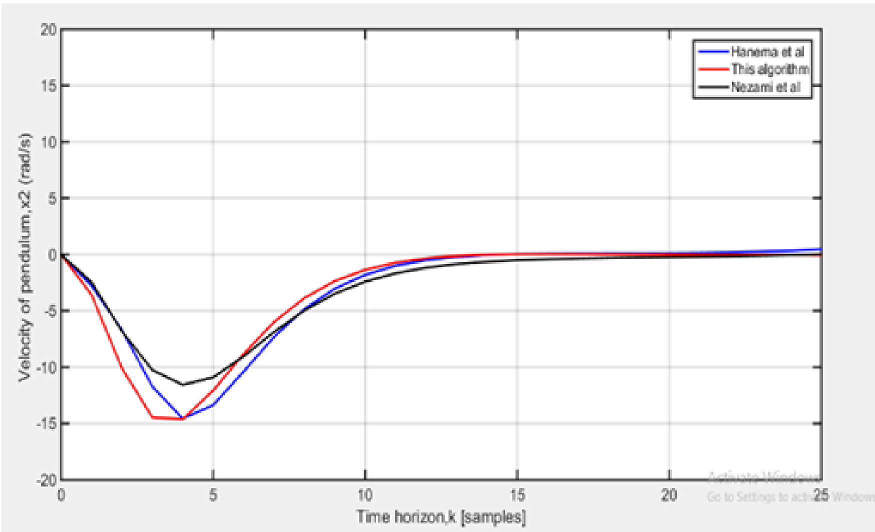

Figure 4: Trajectories for velocity of pendulum of the electrically-driven inverted pendulum

Table 3: Results for motor current of the electrically-driven inverted pendulum

Performance metrics	Controller from Hanema et al. (2021)	Controller from this work	Controller from Nezami et al. (2022)
Response time	0.62 seconds	0.60 seconds	0.92 seconds
Maximum steady state error	0.0247 mA	0.0178 mA	0.1217 mA
Constraints satisfaction	Good	Good	Good
Maximum overshoot/undershoot	2.181 mA	2.152 mA	1.432 mA

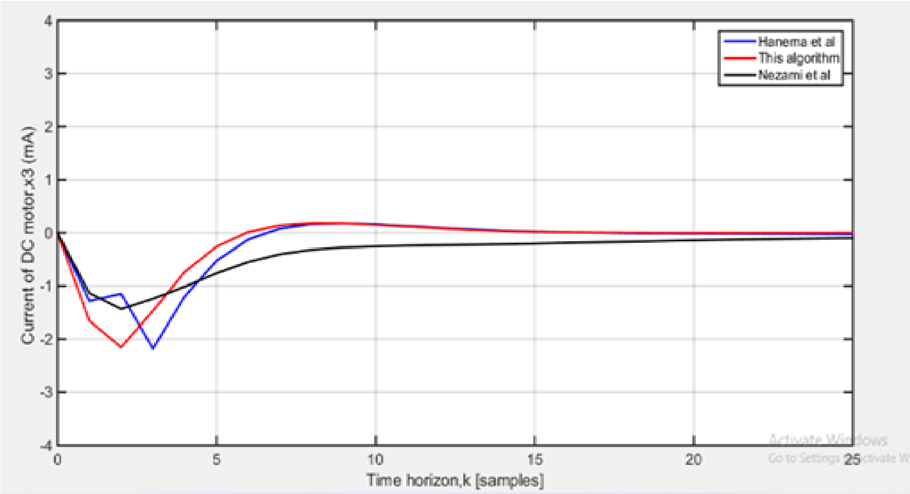

Figure 5: Trajectories for motor current of the electrically-driven inverted pendulum

3.2 Results for the Two-tank System

From the results as shown in Figure 6 and Figure 7, the trajectories of levels – h_1 and h_2 – of the two tank system controlled by the three controllers were able to track effectively their respective set-points without significant steady state error while satisfying the specified state constraints of the system. Meanwhile, all the three controllers produced same pattern of trajectories – causing the trajectories to overlap – with critically damped responses for each of the two states of the two-tank system, which may largely be due to the small measure of nonlinearity in the system.

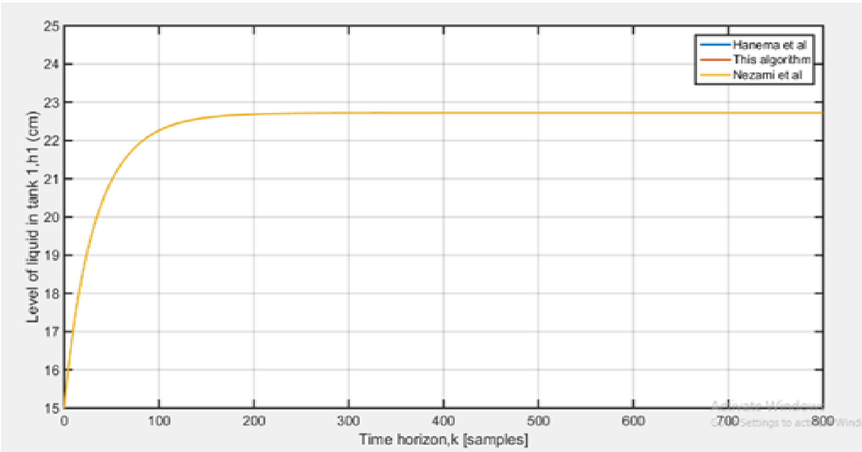

Figure 6: Trajectories for desired level, h_1 of the two-tank system

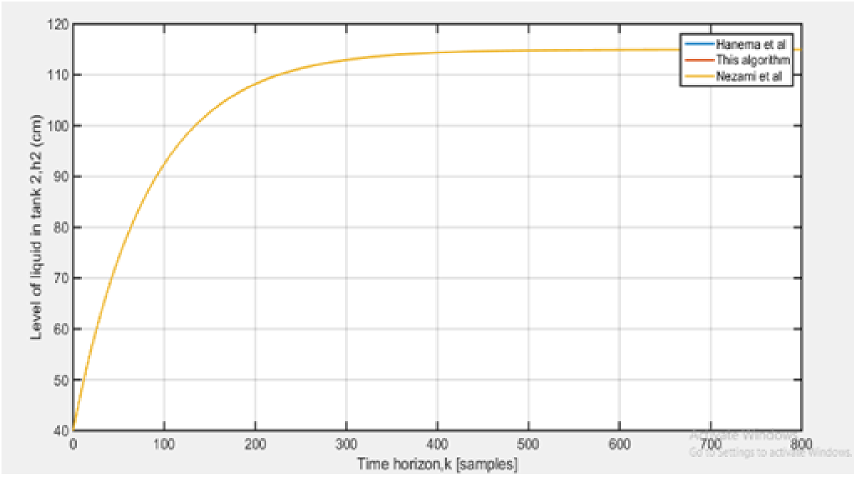

Figure 7: Trajectories for desired level, h_2 of the two-tank system

IV. DISCUSSION AND CONCLUSION

The utilization of Lyapunov function as a terminal cost function in a tube-based model predictive control (TMPC) strategy that employs the use of a contractive terminal set as demonstrated in this paper, helps in mitigating the model predictive controller's restraints which enhances its capacity to cause faster convergence of state trajectories to desired values with a more faithful tracking of the set points, most especially when used to control systems with a high-level of nonlinearity like an electrically driven inverted pendulum. The goal of this work – which is to adequately reduce the conservatism of the tube-based model predictive control (TMPC) method in order to ensure the achievement of control objectives – has been achieved by the results obtained.

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Appendix I

Tube Realization

Parameterization of Cross Sections

Each of the cross section of the tube is represented by $\alpha_i \in \mathbb{R}_+$, being a scaling factor and $z_i \in \mathbb{R}^{n_x}$, being a centre, which are regarded as decision variables. The scaling is governed by:

$$\forall i \in [0 \dots N]: \alpha_i \geq 0. \tag{A.1}$$

Initial Condition Constraints

The constraints $X_0 = \{x\}$ is executed by the constraints

$$z_0 = x,\ \alpha_0 = 0 \tag{A.2}$$

Tube Transition Constraints

The transition constraints for the tubes are described by

$$\forall i \in [0 \dots N-1]: \forall (j,l) \in [1 \dots q_s] \times [1 \dots q_{\theta i}] : H_s(A(\bar{\theta}_i^l)(z_i + \alpha_i \bar{s}^j) + Bu_i^{(j,l)}) \leq 1 \tag{A.3}$$

where $u_i^{(j,l)} \in \mathbb{R}^{n_u}$ are decision variables.

$$\Theta_i = convh\{\bar{\theta}^1, \dots, \bar{\theta}^{q\theta i}\} \text{ and } \mathrm{S} = \mathrm{convh}\{\bar{s}^1, \dots, \bar{s}^{qs}\} = \{x | H_s x \leq 1\}.$$

State Constraints

The state constraints, governed by $X_i \subseteq \mathbb{X}_i$ are described by:

$$\forall i \in [0 \dots N-1]: \forall j \in [1 \dots q_s]: H_{xi}\{z_i + \alpha_i \bar{s}^j\} \leq 1 \tag{A.4}$$

provided $\mathbb{X}_i = \{x | H_{xi} x \leq 1\}$ are present.

Input Constraints

The constraints for the input are governed by

$$\forall i \in [0 \dots N-1]: \forall (j,l) \in [1 \dots q_s] \times [1 \dots q_{\theta i}]: H_u u_i^{(j,l)} \leq 1 \tag{A.5}$$

for a hyperplane classification, $\mathbb{U} = \{u | H_u u \leq 1\}$.

Terminal Constraint

The terminal constraint, described as $X_N \subseteq X_f$ is governed by:

$$\forall j \in [0 \dots N-1]: H_f(z_N + \alpha_N \bar{s}^j) \leq \gamma, 0 \leq \gamma \leq 1, \tag{A.6}$$

with $\gamma \in \mathbb{R}_+$ being a decision variable and $X_f = \{x | H_f \leq 1\}$ being a hyperplane classification.

Terminal Cost

$\gamma \in \mathbb{R}_+$, a decision variable, has been configured to correspond to $\Psi_{X_f}(X_N)$ which allows:

$$F(X_N) = \frac{\gamma}{1-\lambda}\rho \tag{A.7}$$

where $\rho = \sum_{j=1}^{q_s} ||z_N \oplus \alpha_N \bar{s}^j||_{pl}^2$

Full Quadratic Programming using Centre-based Stage Cost

The total QP realizing the tube synthesis problem becomes:

$$\min_d \frac{\gamma}{1-\lambda}\rho + \sum_{i=0}^{N-1}\left(z_i^T Q^T Q z_i + p\alpha_i^2 + \frac{1}{q_{\theta i} q_s}\sum_{j=1}^{q_s}\sum_{l=1}^{q_{\theta i}}\left[u_i^{(j,l)}\right]^T R^T R u_i^{(j,l)}\right) \tag{A.8}$$

subject to (A.1) – (A.6), with d being the decision variables