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Analyticity of Two-dimensional Generalized Canonical Cosine-Cosine Transform

N.V.Kalyankar
Yeshwant Mahavidyalaya, Nanded-431602 (India)

Abstract—In this paper the two-dimensional (2-D) generalized canonical cosine-cosine transform it is extended to the distribution of compact support by using kernel method.

Keywords—2-D fractional Fourier transform,2-D canonical transform, 2-D cosine-cosine transform, generalized function, testing function space.

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I. INTRODUCTION

Now a days fractional Fourier transforms plays important role in signal processing, image reconstruction, pattern recognition, acoustic signal processing, computer data protection.[2],[4],[6].Bhosale and Choudhary [3],had extended fractional Fourier transform to the distribution of compact support. Extensions of some transforms to generalized functions have been done time to time and their properties have been studies by various mathematicians. However, there is much scope in extending double transforms to a certain class of generalized functions. This situation motivated us for extension of double transforms.[1],[5],[7].The two dimensional canonical cosine-cosine transform, given by the integral form is

$$\left\{2DCCCT\, f\left(t,x\right)\right\}\left(s,w\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{c_1}\left(t,s\right)K_{c_2}\left(x,w\right) f\left(t,x\right)dtdx \qquad \ldots\ldots\ldots 1.1$$

Where, $K_{c_1}\left(t,s\right)$ and $K_{c_2}\left(x,w\right)$ are kernels

Notation and terminology of this paper is as per [8], [9].The paper is organized as follows. Section 2 gives definition testing space and the definition of 2-D canonical cosine-cosine transform in section 3 analyticity and uniqueness theorem are proved. Section 4 properties of kernel and modulation theorems are discussed. Lastly the conclusion is stated.

II. DEFINITION OF TESTING FUNCTION SPACE

An infinitely differentiable complex valued function ϕ on R^n belongs to $E(R^n)$, if for each compact set $I \subset s_a$, $J \subset s_b$

where $\quad s_a=\left\{t : t\in R^n,\ \ |t|\le a,\ a>0\right\}$, $s_b=\left\{x : x\in R^n,\ \ |x|\le b,\ b>0\right\}$

and for $I\in R^n$, $J\in R^n$,

$$\gamma_{E,k}\phi\left(t,x\right)=\sup_{\substack{-\infty<t<\infty\\ -\infty<t<\infty}}\left|D_t^{\,k}D_x^{\,l}\,\phi(t,x)\right|<\infty$$

k=0,1,2,3....and l=0,1,2,3

Thus $E\left(R^n\right)$ will denotes the space of all $\phi\left(t,x\right)\in E\left(R^n\right)$ with support contained in s_a and s_b. Note that space E is complete, Frechet space and $E^{'}$ denotes the dual space of E.

2.1 Definition of two Dimensional (2D) Generalized canonical cosine- cosine transform:

Let $E^1(R\,\mathrm{x}\,R)$ denote the dual of $E(R\,\mathrm{x}\,R)$. Therefore the generalized canonical cosine-cosine transform of $f(t,x)\in E^{'}(R\,\mathrm{x}\,R)$ is defined as

$$\left\{2DCCCT\ f(t,x)\right\}(s,w)=\left\langle f(t,x),K_{C_1}(t,s)\ K_{C_2}(x,w)\right\rangle$$

$$\left\{2DCCCT\ f(t,x)\right\}(s,w)$$

$$=\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)\cos\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}\ f(t,x)\,dxdt$$

where, $K_{C_1}(t,x)=\dfrac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\cos\left(\dfrac{s}{b}t\right)$ $\qquad$ when $b\neq 0$

$$= \sqrt{d}\, e^{\frac{i}{2}\left(cds^2\right)} \delta(t-ds) \qquad \text{when } b=0$$

$$\& \; K_{C_2}(x,w) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} . e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} . \cos\left(\frac{w}{b}x\right) \qquad \text{when } b \neq 0$$

$$= \sqrt{d}\, e^{\frac{i}{2}(cdw^2)} \delta(x-dw) \qquad \text{where } b=0$$

$$\gamma_{E,k}\left\{K_{C_1}(t,s)\,K_{C_2}(x,w)\right\} = \sup_{\substack{-\infty<t<\infty \\ -\infty<x<\infty}} \left| D_t^k D_x^l K_{C_1}(t,s) K_{C_2}(x,w) \right| < \infty \; .$$

2.2 Definition of 2-D Canonical sine-sine transform:

$\left\{2DCSST\; f(t,x)\right\}(s,w) =$

$$(-1)\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right) \sin\left(\frac{w}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} . e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt \qquad b \neq 0 \; ,$$

III. ANALYTICITY AND UNIQUENESS THEOREM FOR CANONICAL COSINE-COSINE TRANSFORM

Theorem 3.1 :(Analyticity) Let $f \in E^1(R^n)$ and its two dimensional canonical cosine-cosine transform be defined by,

$$\left\{2DCCCT\; f(t,x)\right\}(s,w)$$
$$= \sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} \cos\left(\frac{s}{b}\right)t \cos\left(\frac{w}{b}\right)x\; f(t,x)\,dxdt$$

then $\left\{2DCCCT\; f(t,x)\right\}(s,w)$ is analytic on C^n ,if the a, b, $\sup pf \subset s_a$ and s_b where $s_a = \left\{t : t \in R^n,\; |t| \le a,\, a>0\right\}$, $s_b = \left\{x : x \in R^n,\; |x| \le b,\, b>0\right\}$ moreover

$\left\{2DCCCT\; f(t,x)\right\}(s,w)$ is differentiable and

$$D_s^k D_w^l \left\{2DCCCT\; f(t,x)\right\}(s,w) = \left\langle f(t,x), D_s^k D_w^l K_{C_1}(t,s) K_{C_2}(x,w)\right\rangle$$

Proof: Let, $s: \{s_1, s_2, \ldots\ldots s_j \ldots\ldots s_n\} \in C^n$ and $w: \{w_1, w_2, \ldots\ldots w_j \ldots\ldots w_n\} \in C^n$.

We first prove that, $\frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j}\{2DCCCT\; f(t,x)\}(s,w)$ exists,

$$\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}\{2DCCCT\; f(t,x)\}(s,w) = \left\langle f(t,x), \frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n} K_{c_1}(t,s) K_{c_2}(x,w)\right\rangle \qquad ....3.1$$

we prove the result $n = 1$, the general result following by induction.

For fixed $s_j \neq 0$ choose two concentric circles C and C^1 with centre s_j and radii r and r_1 respectively such that $0<r<r_1<|s_j|$.

Let Δs_j be a complex increment satisfying $0 < |\Delta s_j| < r$. Also for fixed $w_j \neq 0$.*Again* choose two concentric circles C and C_1 with centre w_j and radii r' and r_1' respectively such that $0< r' < r_1' < |w_j|$.

Let Δw_j be a complex increment satisfying $0 < |\Delta w_j| < r'$

consider, $\frac{(2DCCCT)(s_j + \Delta s_j, w_j) - (2DCCCT)(s_j, w_j)}{\Delta s_j}$.

$$\frac{(2DCCCT)(s_j, w_j + \Delta w_j) - (2DCCCT)(s_j, w_j)}{\Delta w_j} - < f(t,x), \frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j} K_{c_1}(t,s) K_{c_2}(x,w) >$$

$$= \left\langle f(t,x), \Psi\Delta s_j(t)\Psi\Delta w_j(x)\right\rangle \qquad ....3.2$$

where $\Psi\Delta s_j(t)\Delta w_j(x) = \frac{1}{\Delta s_j}\left[K_{c_1}(t, s_1, s_2 \cdots\cdots s_j + \Delta s_j \cdots s_n) - K_{c_1}(t,s)\right]$

$$\frac{1}{\Delta w_j}\left[K_{c_2}(x, w_1, w_2 \cdots w_j + \Delta w_j \cdots w_n) - K_{c_2}(x,w)\right] - \frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n} K_{c_1}(t,s) K_{c_2}(x,w)$$

For any fixed $(t,x) \in R^n$ and any fixed integer.

$k = (k_1, k_2 \cdots\cdots k_n) \in N_0$ and $l = (l_1, l_2 \cdots\cdots l_n) \in N_0$

$D_t^k D_x^l K_{c_1}(t,s) K_{c_2}(x,w)$ is analytic inside and on C' and C'_1.

By Cauchy integral formula.

$D_t^k D_x^l \ \Psi \Delta s_j \Delta w_j (t,x)$

$$= \frac{1}{4\pi^2 i^2} D_t^k D_x^l \int_{c'}\int_{c'_1} K_{s_1}(t,s) K_{s_2}(x,w) \left(\frac{1}{\Delta s_j}\left(\frac{1}{z - s_j - \Delta s_j} - \frac{1}{z - s_j} \right) - \frac{1}{(z - s_j)^2} \right)$$
$$\left(\frac{1}{\Delta w_j}\left(\frac{1}{y - w_j - \Delta w_j} - \frac{1}{y - w_j} \right) - \frac{1}{(y - w_j)^2} \right) dzdy$$

where, $\overline{S} = (s_1 \ldots\ldots s_{j-1}, z, s_{j+1} \ldots\ldots s_n)$ and $\overline{W} = (w_1 \ldots\ldots w_{j-1}, y, w_{j+1} \ldots\ldots w_n)$.

$$= \frac{\Delta s_j \Delta w_j}{-4\pi^2} \int_{c'}\int_{c'_1} \frac{D_t^k D_x^l \ K_{c_1}(t,\overline{s}) K_{c_2}(x,\overline{w})}{(z - s_j - \Delta s_j)(z - s_j)^2 (y - w_j - \Delta w_j)(y - w_j)^2} \, dzdy,$$

But for all $z \in C'$ and $y \in C'_1$ and (t,x) restricted to a compact subset of R^n,

$D_t^k \ D_x^l K_{c_1}(t,s) K_{c_2}(x,w)$ is bounded by constant Q.

$$\left| D_t^k \ D_x^l \Psi \Delta s_j \Delta w_j (t,x) \right| \le \frac{|\Delta s_j||\Delta w_j|}{4\pi^2} \int_{c'}\int_{c'_1} \frac{Q}{(r_1 - r)(r_1)(r'_1 - r)(r'_1)} |dz||dy|$$
$$\le \frac{|\Delta s_j||\Delta w_j|}{4\pi^2} \frac{Q}{(r_1 - r)(r_1)(r'_1 - r)(r'_1)}$$

Thus as $|\Delta s_j| \to 0,$ and $|\Delta w_j| \to 0,$ $D_t^k \ D_x^l \Psi \Delta s_j \Delta w_j (t,x)$ tends to zero uniformly on the compact subset of R^n,therefore it follows that $\Psi \Delta s_j \Delta w_j (t,x)$ converges in $E(R^n)$ to zero. Since $f \in E^1$, we conclude (3.2) tends to zero. Therefore $\{2DCCCT \ f(t,x)\}(s,w)$ is differentiable with respective s_j and w_j. But this is true for all i, $j=1,2,\ldots\ldots n$ .Hence by induction method we can easily show that $\{2DCCCTf(t,x)\}(s,w)$ is infinitely differentiable

$\{2DCCCTf(t,x)\}(s,w)$ is analytic on C^n.

and $D_s^k D_w^l \{2DCCCT \ f(t,x)\}(s,w) = \left\langle f(t,x), D_s^k D_w^l \ K_{c_1}(t,s) K_{c_2}(x,w) \right\rangle$.

Theorem4.2 :(Uniqueness) If $\{2DCCCT \ f(t,x)\}(s,w)$ and $\{2DCCCT \ g(t,x)\}(s,w)$ are 2D canonical cosine-cosine transform and

$\sup pf \subset s_a$, and s_b and, $\sup p \ g \subset s_a$, and s_b

Where $s_a = \{t : t \in R^n, \ |t| \le a, a > 0\}$ and $s_b = \{x : x \in R^n, \ |x| \le b, b > 0\}$

If $\{2DCCCT \ f(t,x)\}(s,w) = \{2DCCCT \ g(t,x)\}(s,w)$

then, $f = g$ in the sense of equality in $D'(I)$

Proof: By inversion theorem

$$\therefore f - g = \sqrt{\frac{2\pi i}{b}} \sqrt{\frac{2\pi i}{b}} e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-\frac{i}{2}\left(\frac{a}{b}\right)x^2} \int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos\left(\frac{s}{b}x\right)\cos\left(\frac{w}{b}x\right)$$
$$\left[\{2DCCCTf(t,x)\} - \{2DCCCTg(t,x)\}\right] dsdw$$

Thus $f = g$ in $D'(I)$

IV. PROPERTIES OF KERNEL

Kernel of 2D canonical cosine- cosine transforms satisfied following properties.

(1) $K_{C_1}(t,s) \ K_{C_2}(x,w) = K_{C_1}(s,t) \ K_{C_2}(w,x)$ if $a = d$

(2) $K_{C_1}(t,s) \ K_{C_2}(x,w) \ne K_{C_1}(s,t) \ K_{C_2}(w,x)$ if $a \ne d$

(3) $K_{C_1}(-t,s) \ K_{C_2}(x,w) = K_{C_1}(t-s) \ K_{C_2}(x,w)$

(4) $K_{C_1}(-t,s) \ K_{C_2}(-x,w) = K_{C_1}(t,s) \ K_{C_2}(x,w)$

4.1. Modulation theorems for canonical cosine-cosine transform:

Theorem4.1.1: If $\{2DCCCT\ f(t,x)\}(s,w)$ is canonical cosine-cosine transform of $f(t,x)$ then

$$\{2DCCCT\cos\mu t\ f(t,x)\}(s,w)=$$

$$\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{2DCCCT\ f(t,x)\}(s+\mu b,w)+e^{i(\mu sd)}\{2DCCCT\ f(t,x)\}(s-\mu b,w)\right]$$

Theorem 4.1.2: If $\{2DCCCT\ f(t,x)\}(s,w)$ is canonical cosine-cosine transform of $f(t,x)$ then

$$\{2DCCCT\sin\mu t\ f(t,x)\}(s,w)$$

$$=\frac{ie^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{2DCSCT\ f(t,x)\}(s+\mu b,w)-e^{i(\mu sd)}\{2DCSCT\ f(t,x)\}(s-\mu b,w)\right]$$

Theorem 4.1.3: If $\{2DCCCT\ f(t,x)\}(s,w)$ is canonical cosine-cosine transform of $f(t,x)$ then

$$\{2DCCCTe^{i\mu t}\ f(t,x)\}(s,w)=\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\Big[e^{-i(s\mu d)}\left(\{2DCCT\ f(t,x)\}(s+\mu b,w)-\{2DCSCT\ f(t,x)\}(s+\mu b,w)\right)$$
$$+e^{i(s\mu d)}\left(\{2DCCT\ f(t,x)\}(s-\mu b,w)+\{2DCSCT\ f(t,x)\}(s-\mu b,w)\right)\Big]$$

V. CONCLUSION

In this paper two-dimensional canonical cosine-cosine is generalized in the form the distributional sense, analyticity and uniqueness theorems for this transform are proved. Some properties of kernel and modulation theorems are discussed.

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