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Coupled Fixed Point Theorem in Ordered Metric Spaces

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Abstract:- In this paper we prove a coupled fixed point theorem satisfying a new type of contractive conditions by using the concept of g- monotone mapping in ordered metric space. Our result is generalization of previous coupled fixed point theorem.
Keywords:- Coupled fixed point, Coupled Common Fixed Point, Mixed monotone, Mixed g- monotone mapping.

I. INTRODUCTION

The Banach contraction principle is the most celebrated fixed point theorem. Boyd and Wong [4] extended the Banach contradiction principle to the case of non linear contraction mappings. Afterword many authors obtain important fixed point theorems. Recently Bhaskar and Lakshmikantham [2] presented some new results for contractions in partially ordered metric spaces, and noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary valued problem.

After some time, Lakshmikantham and Circ [6] introduced the concept of mixed monotone mapping and generalized the results of Bhaskar and Lakshmikantham [2]. In the present work, we prove some more results for coupled fixed point theorems by using the concept of g-monotone mappings.

Recall that if $(X, \leq)$ is partially ordered set and $F : X \to X$ such that for $x, y \in X, x \leq y$ implies $F(x) \leq F(y)$ then a mapping F is said to be non decreasing. similarly, mapping is defined. Bhaskar and Lakshmikantham introduced the following notions of mixed monotone mapping and a coupled fixed point.

Definition- 1 (Bhaskar and Lakshmikantham [2]) Let $(X, \leq)$ be an ordered set and $F : X \times X \to X$. The mapping F is said to has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(y_2, x) \leq F(y_1, x)$$

Definition – 2 (Bhaskar and Lakshmikantham [2]) An element $(x, y) \in X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if,

$$F(x, y) = x, \; F(y, x) = y$$

The main theoretical results of the Bhaskar and Lakshmikantham in [2] are the following two coupled fixed point theorems.

Theorem – 3 (Bhaskar and Lakshmikantham [2] Theorem 2.1) Let $(X, \leq)$ be partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with ,

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)],$$

for each, $x \geq u$, and, $y \leq v$
If there exist $x_0, y_0 \in X$ such that,

$$x_0 \leq F(x_0, y_0), \text{and}, y_0 \geq F(y_0, x_0)$$

Then there exists, $x, y \in X$ such that,

$$x = F(x, y), \text{and}, \; y = F(y, x)$$

Theorem – 4 (Bhaskar and Lakshmikantham [2] Theorem 2.2) Let $(X, \leq)$ be partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property,

i. If a non decreasing sequence $\{x_n\} \leq x$, then $x_n \leq x$, for all n,
ii. If a non increasing sequence $\{y_n\} \leq y$, then $y \leq y_n$, for all n,

Let F : X \times X \rightarrow X be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with ,

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)],$$

for each, $x \geq u$, and, $y \leq v$
If there exist $x_0, y_0 \in X$ such that,

$$x_0 \leq F(x_0, y_0) \text{ ,and}, y_0 \geq F(y_0, x_0)$$

Then there exists, $x, y \in X$ such that,

$$x = F(x,y), \text{and}, \ y = F(y,x)$$

We note that Bhaskar and Lakshmikantham [2] have discussed the problems of a uniqueness of a coupled fixed point and applied their theorems to problems of the existence and uniqueness of solution for a periodic boundary valued problem.

II. MAIN RESULTS

Analogous with Definition 2 Lakshmikantham and Ciric [6] introduced the following concept of a mixed g-monotone mapping.

Definition – 5 Let $(X, \leq)$ be an ordered set and $F : X \times X \to X$ and $g: X \times X$. The mapping F is said to has the mixed g-monotone property if F is g- monotone non-decreasing in its first argument and is g- monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(y_2, x) \leq F(y_1, x)$$

Note that if g is the identity mapping, then Definition – 5 , reduces to Definition – 2 .

Definition – 6 (Lakshmikantham and Ciric [6]) An element $(x, y) \in X$ is called a coupled coincidence point of the mapping $F: X \times X \to X$ and $g: X \to X$ if,

$$F(x,y) = g(x) \ , \ F(y,x) = g(y)$$

Definition – 7 (Lakshmikantham and Ciric [6])Let X be non empty set and $F: X \times X \to X$ and $g: X \to X$ one says F and g are commutative if,

$$g(F(x,y)) = F(g(x), g(y)) \ ,$$

for all $x, y \in X$.

Now we prove our main result.

Theorem – 8 Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F: X \times X \to X$ and $g: X \to X$ are such that F has the mixed g- monotone property and

$$d(F(x,y), F(u,v)) \leq p \ \max\{ d(g(x), F(x,y)), d(g(u), F(u,v))\} + q \max\{ d(g(x), F(u,v)), d(g(u), F(x,y))\} \quad (1)$$

for all $x, y, u, v \in X$ and $p, q \in [0,1)$ such that $0 \leq p + q < 1$ with $g(x) \geq g(u)$ and $g(y) \leq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commute with F and also suppose either

i. If a non decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all n,

ii. If a non increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, for all n.

If there exist $x_0 , y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(x_0, y_0)$$

Then there exist $x, y \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(x_0, y_0)$. Since

$$g(x) = F(x,y) \text{ and } g(y) = F(y,x).$$

Proof: Let $x_0, y_0 \in X$ be such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(x_0, y_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0 , x_0)$. Again from $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing the process we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \quad (2)$$

for all $n \geq 0$.

We shall show that

$$g(x_n) \leq g(x_{n+1}) \quad \text{for all } n \geq 0. \quad (3)$$

and

$$g(y_n) \geq g(y_{n+1}) \quad \text{for all } n \geq 0 \quad (4)$$

For this we shall use the mathematical induction. Let n = 0. Since $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(x_0, y_0)$ and as $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$. Thus (3) and (4) hold for n = 0.

Suppose now (3) and (4) holds for some fixed $n \geq 0$. Then, since $g(x_n) \leq g(x_{n+1})$ and $g(y_{n+1}) \geq g(y_n)$, and as F has the mixed g- monotone property, from (3.4) and (2.1).

$$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n) \quad (5)$$

And

$$F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1}) \quad (6)$$

and from (3.4) and (2.2),

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n) \quad (7)$$

and

$$F(y_{n+1}, x_n) \geq F(y_{n+1}, x_n) = g(y_{n+2}) \quad (8)$$

Now from (5) – (8) we get

$$g(x_{n+1}) \leq g(x_{n+2}) \quad (9)$$

and

$$g(y_{n+1}) \geq g(y_{n+2}) \quad (10)$$

Thus by the mathematical induction we conclude that (5) – (8) holds for all $n \geq 0$. Therefore,

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq g(x_3) \leq \ldots\ldots\ldots\ldots \leq g(x_n) \leq g(x_{n+1}) \leq \ldots\ldots$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq g(y_3) \geq \ldots\ldots\ldots\ldots \geq g(y_n) \geq g(y_{n+1}) \geq \ldots\ldots$$

Since, $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$

$$d\big(g(x_n), g(x_{n+1})\big) = d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big)$$

by using, (1) and (2) we have,

$$d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \leq p \max\{ d\big(g(x_{n-1}), F(x_{n-1}, y_{n-1})\big), d\big(g(x_n), F(x_n, y_n)\big)\} + q \max\{ d\big(g(x_n), F(x_{n-1}, y_{n-1})\big), d\big(g(x_{n-1}), F(x_n, y_n)\big)\}$$

This gives,

$$d\big(g(x_n), g(x_{n+1})\big) \leq p \max\{ d\big(g(x_{n-1}), g(x_n)\big), d\big(g(x_n), g(x_{n+1})\big)\} + q \max\{ d\big(g(x_n), g(x_n)\big), d\big(g(x_{n-1}), g(x_{n+1})\big)\}$$

$$d(g(x_n), g(x_{n+1}) \leq \frac{p+q}{1-q} d\big(g(x_{n-1}), g(x_n)\big) \quad (11)$$

Similarly, from (1) and (2), as $g(y_n) \leq g(y_{n-1})$ and $g(x_n) \geq g(x_{n-1})$

$$d\big(g(y_{n+1}), g(y_n)\big) = d\big(F(y_n, x_n), F(y_{n-1}, x_{n-1})\big)$$

$$d\big(F(y_n, x_n), F(y_{n-1}, x_{n-1})\big) \leq p \max\{ d\big(g(y_n), F(y_n, x_n)\big), d\big(g(y_{n-1}), F(y_{n-1}, x_{n-1})\big)\} + q \max\{ d\big(g(y_n), F(y_{n-1}, x_{n-1})\big), d\big(g(y_{n-1}), F(y_n, x_n)\big)\}$$

$$d\big(g(y_{n+1}), g(y_n)\big) \leq p \max\{ d\big(g(y_n), g(y_{n+1})\big), d\big(g(y_{n-1}), g(y_n)\big)\} + q \max\{ d\big(g(y_n), g(y_n)\big), d\big(g(y_{n-1}), g(y_{n+1})\big)\}$$

$$d(g(y_{n+1}), g(y_n) \leq \frac{p+q}{1-q} d\big(g(y_{n-1}), g(y_n)\big) \quad (12)$$

Let us denote $\frac{p+q}{1-q} = h$ and,

$$d\big(g(x_n), g(x_{n+1})\big) + d\big(g(y_{n+1}), g(y_n)\big) = d_n$$

then by adding (11) and (12), we get

$$d_n \leq hd_{n-1} \leq h^2 d_{n-2} \leq h^3 d_{n-3} \leq \ldots\ldots \leq h^n d_0$$

which implies that,

$$\lim_{n\to\infty} d_n = 0$$

Thus,

$$\lim_{n\to\infty} d\big(g(x_{n+1}), g(x_n)\big) = \lim_{n\to\infty} d\big(g(y_{n+1}), g(y_n)\big) = 0$$

For each $m \geq n$ we have,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + \ldots\ldots + d(x_{m-1}, x_m)$$

and

$$d\big(g(y_n), g(y_m)\big) \leq d\big(g(y_n), g(y_{n+1})\big) + d\big(g(y_n), g(y_{n+2})\big) + \ldots\ldots + d\big(g(y_{m-1}), g(y_m)\big)$$

by adding the both of above, we get

$$d\big(g(x_n), g(x_m)\big) + d\big(g(y_n), g(y_m)\big) \leq \frac{h^n}{1-h} d_0$$

which implies,

$$\lim_{n\to\infty} \Big(d\Big(g(x_{\{n\}}), g(x_m)\Big) + d\big(g(y_n), g(y_m)\big)\Big) = 0$$

Therefore, $\{ g(x_n)\}$ and $\{ g(y_n)\}$ are Cauchy sequence in X. since X is complete metric space, there exist $x, y \in X$ such that $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$.

Thus by taking limit as $n \to \infty$ in (2), we get

$$x = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F(x, y)$$

$$y = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(y, x)$$

Therefore, F and g have a coupled fixed point.

Now, we present coupled coincidence and coupled common fixed point results for mappings satisfying contractions of integral type. Denote by Λ the set of functions $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

i. α is a Lebsegue mapping on each compact subet of $[0, +\infty)$,

ii. for any $\epsilon > 0$, we have $\int_0^{\epsilon} \alpha(s)ds > 0$.

Finally we have the following results.

Theorem – 9 Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F: X \times X \to X$ and $g: X \to X$ are such that F has the mixed g- monotone property and assume that there exist $\alpha \in \Lambda$ such that,

$$\int_0^{d^2(F(x,y),F(u,v))} \alpha(s)ds \leq p \int_0^{d(g(x),F(x,y)).d(g(u),F(u,v))} \alpha(s)ds + q \int_0^{d(g(u),F(x,y)).d(g(x),F(u,v))} \alpha(s)ds$$

for all $x, y, u, v \in X$ and $p, q \in [0,1)$ such that $0 \leq p + q < 1$ with $g(x) \geq g(u)$ and $g(y) \leq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commute with F. Then there exist $x, y \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x)$$

In the view of Theorem – 8 , we have,

$$d(g(x_n), g(x_{n+1})) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))$$

By using, (3.14) we have,

$$\int_0^{d^2(F(x_{n-1},y_{n-1}),F(x_n,y_n))} \alpha(s)ds \le p \int_0^{d(g(x_{n-1}),F(x_{n-1},y_{n-1})).d(g(x_n),F(x_n,y_n))} \alpha(s)ds$$
$$+ q \int_0^{d(g(x_n),F(x_{n-1},y_{n-1})).d(g(x_{n-1}),F(x_n,y_n))} \alpha(s)ds$$
$$\int_0^{d^2(g(x_n),g(x_{n+1}))} \alpha(s)ds \le \int_0^{d(g(x_{n-1}),g(x_n)).d(g(x_n),g(x_{n+1}))} \alpha(s)ds$$
$$\int_0^{d(g(x_n),g(x_{n+1}))} \alpha(s)ds \le p \int_0^{d(g(x_{n-1}),g(x_n))} \alpha(s)ds$$

It can be written as,

$$d(g(x_n), g(x_{n+1})) \le p\ d(g(x_{n-1}), g(x_n))$$

Similarly we can show that,

$$d(g(x_n), g(x_{n+1})) \le p^2\ d(g(x_{n-2}), g(x_{n-1}))$$

Processing the same way it is easy to see that,

$$d(g(x_n), g(x_{n+1})) \le p^n\ d(g(x_0), g(x_1))$$

and

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0$$

From the Theorem – 8 , the result is follows and nothing to prove.

Corollary – 10 Let $(X, \le)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F: X \times X \to X$ and $g: X \to X$ are such that F has the mixed g- monotone property such that,

$$d^2(F(x,y), F(u,v)) \le p\ (d(g(x), F(x,y)).\ d(g(u), F(u,v)))$$
$$+ q\ \left(d(g(u), F(x,y)).\ d(g(x), F(u,v))\right)$$

for all $x, y, u, v \in X$ and $p, q \in [0,1)$ such that $0 \le p + q < 1$ with $g(x) \ge g(u)$ and $g(y) \le g(v)$. Suppose that $F(X \times X)$\subseteq $g(X)$, g is continuous and commute with F. Then there exist $x, y \in X$ such that $g(x_0) \le F(x_0, y_0)$ and $g(y_0) \ge F(x_0, y_0)$. Since

$$g(x) = F(x,y) \text{ and } g(y) = F(y,x) .$$

Proof : In Theorem – 9 , if we take $\alpha(s) = 1$ then result is follows.

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