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# {2, 2}-Extendability of Planar Graphs

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**Abstract:**- In this paper, the idea of assigning lists of varying sizes to vertices of a planar graph will be explored. Thomassen's 5-list-coloring theorem states that plane graphs are list-colorable when two adjacent vertices on the boundary of the unbounded face are precolored, other vertices on the boundary of the unbounded face are assigned lists of size 3, and all other vertices of the graph are assigned lists of size 5. This can be thought of as being 2-extendable. Thomassen also defined an analogous property of 3-extendability, and later Definition , which corresponds to having the vertices of a 3-path along the boundary of the unbounded face precolored. While every planar graph is 2-extendable, it is not the case that every planar graph is 3-extendable.

**Keywords:**- {i, j}-extendability, {2, 2, 2}-extendable, 3-cycle.

## I. INTRODUCTION

The following section will describe this notion in more detail.
Hutchinson defines the following notion of {i, j}-extendability.

**Definition 4.1.** Let G = (V, E) be a plane graph and let C be the cycle that corresponds to the boundary of the unbounded face of G. Let x, y $\in$ V (C ) be two nonadjacent vertices of C. Let L : V $\rightarrow$ $2^N$ be an arbitrary assignment of lists of colors to the vertices of G such that |L(x)| = i, |L(y)| = j, |L(v)| = 3 for all v $\in$ V (C) - {x, y}, and |L(w )| = 5 for all w $\in$ V - V (C). If G is L-colorable for all such list assignments L, then G is said to be (i, j )-extendable with respect to (x, y). If G is (i,j )-extendable with respect to (x, y) for every pair of vertices x, y $\in$ V (C ), then G is said to be {i, j}-extendable.

Hutchinson characterized all {1, 1}- and {1, 2}-extendable outerplanar graphs and showed that every outerplanar graph is {2,2}-extendable. Here an alternate proof of the {2, 2}- extendability of outerplanar graphs will be presented and the following conjecture, posed by Hutchinson, will be explored.

**Conjecture .** Plane graphs are {2, 2}-extendable.

This chapter contains results that provide some types of planar graphs that are {2, 2}-extendable.

Let x, y be vertices on the boundary of the unbounded face of a plane graph G, where C is the cycle that corresponds to the boundary of the unbounded face of G. Let T be the set of endpoints of all chords in G. The induced subgraph G[T $\cup$ {x, y}] is said to be an {x, y}-skeleton if it is a tree; in this case it is said that G contains an {x, y}-skeleton. See Figure 4.1 for an example, where the {x, y}-skeleton is shown in bold.

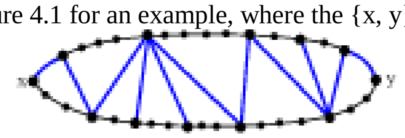



Figure 4.1: $G[C]$ and the corresponding $\{x, y\}$-skeleton $G[T \cup \{x, y\}]$ of $G$.

**Theorem .** Let G be a plane graph, let C = $x_1$ $x_2$ ..... $x_k x_1$ be the cycle that corresponds to the boundary of the unbounded face of G. Let x = $x_1$ and y = $x_j$ for some j$\in$ {2, . . . , k}. Let $\tilde{G}$ be a Type II reduced graph of G and let $\tilde{C}$ be the cycle that corresponds to the boundary of the unbounded face of ˜ G. If one of the following holds:

1. the distance between x and y in $\tilde{G}$ [V ($\tilde{C}$ )] is at most 3, or
2. $\tilde{G}$ contains an {x, y}-skeleton,

then G is (2, 2)-extendable with respect to (x, y).

Note that Theorem 4.3 (1) implies the following corollary. This follows because if the unbounded face of G has at most six vertices, then the distance between x and y in G[V (C)] is at most 3.

**Corollary .** Let G be a plane graph and let $\tilde{G}$ be a Type II reduced graph of G. Let $\tilde{C}$ be the cycle that

corresponds to the boundary of the unbounded face of $\tilde{G}$ . If $|V(\tilde{C})| = 6$, then G is {2,2}-extendable.
**Theorem .** Outer plane graphs and wheels are {2, 2}-extendable.

Let {2, 2, 2}-extendable be defined analogously to {2, 2}-extendable, except three vertices instead of two vertices on the boundary of the unbounded face are assigned lists of size 2. Note that if Conjecture 4.2 is true, the result cannot be strengthened without additional restrictions. This is because not all planar graphs are {1,2}-extendable, see Figures 4.2a and 4.2c , and not all planar graphs are {2, 2, 2}-extendable, see Figures 4.2b and 4.2d , even if the vertices with lists of size 2 are arbitrarily far apart. Note also that the graphs in Figures 4.2a and 4.2c and Figures 4.2b and 4.2d belong to infinite families of planar graphs which are not {1, 2}-extendable or {2,2, 2}-extendable, respectively. These graphs must be such that the lengths of the paths along the boundary of the unbounded face between the vertices with lists of size smaller than 3 must be congruent to 2 mod 3 to get a graph that is not {1, 2}-extendable, and the lengths of the paths along the boundary of the unbounded face between the vertices with lists of size 2 and the inner triangle must be congruent to 1 mod 3 to get a graph that is not {2, 2, 2}-extendable when assigning lists of these types. Recall from earlier that Hutchinson [38] classified all {1, 2}-extendable outer planar graphs, so it was already known that not all planar graphs are {1, 2}-extendable.

## II. PRELIMINARIES

This section contains some previously known results and Section contains some new results that will be used in proving the main theorems of this chapter. It is a known result proven by Erdos et al. and Borodin that a graph G is list-colorable if the size of the list assigned to a vertex is at least the degree of that vertex for each vertex in G, unless G is a Gallai tree and the lists have special properties.

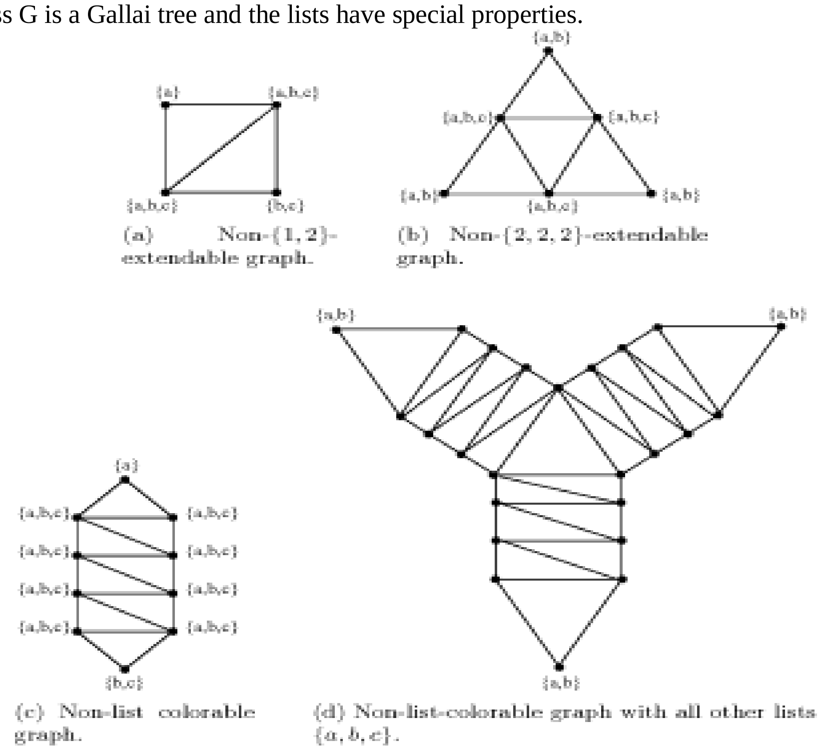
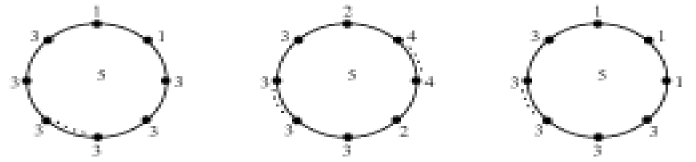



Figure 4.2: Non-extendable graphs.

The following is a result of Bohme, Mohar and Stiebitz which gives a weaker version of {2, 2}-extendability for planar graphs, where the lists are of size 4 along a path of the unbounded face. See Figure 4.3b for a reference to the list sizes in this theorem.

**Theorem** (Bohme et al. ). Let G = (V, E) be a plane graph, let C be cycle that corresponds to the boundary of the unbounded face of G, and let $P = v_1v_2 \ldots\ldots v_{k-1}\, v_k$ be a subpath of C. Let $L : V \rightarrow 2^N$ be an assignment of lists of colors to the vertices of G such that N $|L(v_i)| = 2$ for i = 1, k; $|L(v_i)| = 4$ for all $i \in \{2, \ldots, k-1\}$; $|L(v)| = 3$ for all $v \in V(C)-V(P)$; and $|L(w)| = 5$ for al l $w \in V(G) - V(C)$. Then G is L-colorable.
It is also known that if all of the vertices on a small face of a plane graph are precolored, then it is extendable to a 5-list-coloring of the graph. This result is stated more precisely in the following theorem.



(a) Theorem 1.13. (b) Theorem 4.6. (c) Theorem 4.9.

Figure 4.3: List sizes that indicate $L$-colorability.

**Theorem** (Thomassen ). Let G = (V, E) be a plane graph and let C = $v_1v_2$…. $v_kv_1$1 be the cycle that corresponds to the boundary of the unbounded face of G. Assume k ≤ 5. Let L : V → $2^N$ be an assignment of lists of colors to the vertices of G such that $|L(v_i)| = 1$ for all i = 1,....., k and |L(v)| = 5 for al l v ∈ V - V (C). If G[V(C )] is L-colorable, then G is L-colorable unless k=5, $L(v_i)$ is distinct for each i =1,...,5 and there is a vertex u ∈ V-V (C) such that u ~ $v_i$ for i = 1, …. , 5 and L(u) = $L(v_1) \bigcup \ldots \bigcup L(v_5)$.

**Definition .** Let G = (V, E) be a planar graph and C = $v_1v_2$ …. $v_k$ $v_1$ be the cycle corresponding to one face of G. It is said that G is 3-extendable with respect to the path $v_k$ $v_1$ $v_2$ if G is L-colorable for any assignment L of lists of colors to the vertices of G in which $|L(v_i)| = 1$ for i = 1, 2, k; $|L(v_i)| = 3$ for i = 3, .. . , k - 1; |L(v)| = 5 for v V - V (C), and G[$v_k$, $v_1$, $v_2$ ] is L-colorable.

As described in the following theorem, Thomassen showed that a planar graph G is 3- extendable provided it does not have a subgraph that is a generalized wheel for which the boundary of its unbounded face is made up of vertices that lie on the boundary of the unbounded face of G. See Figure 4.3c for a reference to the list sizes in the following theorem.

**Theorem** (Thomassen ). Let G be a near-triangulation and C = $v_1$ $v_2$ ….. $v_k$ $v_1$ be the cycle that corresponds to the boundary of the unbounded face of G. Then G is 3-extendable with respect to $v_k$ $v_1$ $v_2$ unless there is a subgraph G' of G that is a generalized wheel with principal path $v_1$ $v_2$ $v_k$ and all other vertices that lie on the boundary of the unbounded face of G are elements of V (C). Furthermore, if such a subgraph G exists and G is not a broken wheel, then for each list assignment L, there is at most one proper coloring of G[$v_1$, $v_2$, $v_k$ ] for which G is not 3-extendable with respect to $v_k$ $v_1$ $v_2$.

Assume ($c_k$, $c_1$, $c_2$ ) is the unique proper precoloring of $v_k$ $v_1$ $v_2$ that is not 3-extendable, given that the obstruction is not a broken wheel. Call the triple ($c_k$, $c_1$, $c_2$ ) the bad coloring of $v_k$ $v_1$ $v_2$ with respect to (G, L) and call $c_i$ the bad color of $v_i$, for i = 1, 2, k, with respect to the corresponding bad coloring of $v_k$ $v_1$ $v_2$ , G, and L. For convenience, given a path P = $v_k$ $v_1$ $v_2$ , let $C_P = C_{v_kv_1v_2} = C_{v_kv_1v_2}$ (G, L) denote the ordered triple that is the bad coloring of $v_k$ $v_1$ $v_2$ with respect to (G, L).

**Definition .** It is said that a coloring c of P avoids $C_P$ if, given P = $v_k$ $v_1$ $v_2$ and $C_P$ =
(c , c , c ), then c(v ) ≠ $c_i$ for some i ∈ {1,2, k}. Additionally, it is said that, for some i ∈ {1, 2, k}, a color c of $v_i$ avoids $C_P$ if $c(v_i) \neq c$ .

If G is an odd wheel, then Figure 4.4 illustrates the list assignment that corresponds to the bad coloring of $v_k$ $v_1$ $v_2$ that is not 3-extendable for $W_5$. This list assignment may be generalized for any odd wheel by assigning the list {a, d, e} to any additional vertices.

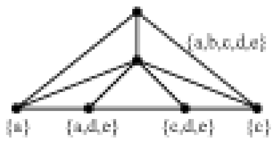



Figure 4.4: Unique non-3-extendable precoloring of an odd wheel.

## 4.3 New results

**Lemma .** Consider a triangle ($x_1$, $x_2$, $x_3$) with lists L of sizes 2, 3, 3 assigned to $x_1$, $x_2$, $x_3$ respectively. Then there are at least three L-colorings of this triangle such that the ordered pairs of colors assigned to $x_2$, $x_3$ are distinct.

**Proof.** Let G be the triangle ($x_1$, $x_2$, $x_3$ ) and assume L($x_1$ ) = {α, β}. Let

S = {(c($x_1$), c($x_2$), c($x_3$)) : c is a proper L-coloring of G}.

Let { γ, γ} ⊆ L($x_2$) - {β}, then S = {(β, γ, q) : q ∈ L($x_3$) - {β, γ }} {(β, γ, q) : q ∈ L(x ) - {β, γ}} ⊆ S. Note here that γ or γ can be a. If |S | ≥ 3, the lemma follows. Otherwise, |L($x_3$) - {β, γ }| = 1 and |L($x_3$) - {β, γ'}| = 1 implying that L($x_3$) = {ß, γ , γ' }. Without loss of generality, assume γ {γ ,γ' } - {α}. Then S {(β, γ , γ' ), (β, γ , γ' ), (α, γ , β)} and the lemma follows.

**Lemma .** Let (u, v,w) be a triangle. Assume there are three distinct ordered pairs of colors ($a_i$, $b_i$ ) for i = 1, 2, 3 that can be assigned to the vertices w, u. If v is assigned a list L(v) of three colors, then there are at least three distinct ordered pairs of colors ($b_i$,$c_i$), i = 1,2, 3, that can be assigned to the vertices u,v for which $c_i \in L(v) - \{a_i, b_i\}$ for i = 1,2, 3.

**Proof.** It is not hard to see that there exists $c_i \in L(v) - \{a_i, b_i\}$ for i = 1, 2, 3. Consider the pairs $(b_1, c_1)$, $(b_2, c_2)$, $(b_3, c_3)$. It remains to show that these three pairs are distinct. Assume $|L(v) - \{a_i,b_i\}| = 1$ for all i = 1, 2, 3. Otherwise, there are more than three pairs and the result follows more easily. Without loss of generality, assume $b_1 = b_2$ and $c_1 = c_2$. This implies two of the pairs for w, u are actually $(a_1, b_1)$, $(a_2, b_1)$. It then follows that $L(v) - \{a_2, b_1\} = c_1 = L(v) - \{a_2, b_1\}$, hence $a_1 = a_2$. This is a contradiction, as it was assumed that $(a_1, b_1)$ and $(a_2, b_2)$ are distinct pairs. Thus, the pairs $(b_1, c_1)$, $(b_2, c_2)$, $(b_3, c_3)$ are distinct and the lemma follows.

The following lemma will be used to show that outerplane graphs are {2, 2}-extendable.

**Lemma.** Let G be an outerplane near-triangulation with vertex x of degree 2. Let L be an assignment of lists of colors to the vertices of G such that $|L(x)| = 2$ and $|L(w)| = 3$ for all $w \in V(G) - \{x\}$. For any edge uv on the unbounded face of G, there are at least three L-colorings of G such that the ordered pairs of colors assigned to u, v are distinct.

**Proof.** If $x \in \{u, v\}$, then there are at least three distinct proper colorings of G[{u, v}] that are each extendable to L-colorings of G by Theorem 1.13 . Thus, assume $x \notin \{u, v\}$.

The proof is by induction on |V (G)|. If G has three vertices, the result follows from Lemma 4.11.

Before proceeding, it can be assumed that G does not contain any non-{x, u, v}-separating chords, otherwise the lemma follows by induction and Theorem 1.13.

Assume the result holds for all outerplane near-triangulations on less than n vertices with list assignments as described in the hypotheses of the lemma. Now consider an outerplane graph G such that |V (G)| = n and choose an arbitrary edge uv on the unbounded face of G for which $x \notin \{u,v\}$. Since G is a near-triangulation and there is no non-{x,u,v}-separating chord in G, there is a $w \in V(G)$ such that (u, v, w) is a triangle in G and either vw is an edge on the unbounded face of G or uw is an edge on the unbounded face of G. Without loss of generality, assume vw is an edge on the unbounded face of G. Consider the graph G - v with lists L. Since |V (G - v)| = n - 1 and uw is an edge on the unbounded face of G - v, there are at least three L-colorings of G - v for which the ordered pairs of colors assigned to w, u are distinct by induction. Let these pairs be $(a_1, b_1)$, $(a_2, b_2)$, $(a_3, b_3)$ where $a_i$ is the color from L(w) assigned to w and $b_i$ is the color from L(u) assigned to u for i = 1, 2, 3.i

The result then follows by Lemma 4.12 .

In Theorem 4.3, the idea of a reduced graph is used. The following lemma illustrates why this notion is helpful.

**Lemma .** Let G be a plane graph, $x, y \in V(G)$ be vertices on the unbounded face C of G, and L an assignment of lists of colors to the vertices of G such that $|L(x)| = |L(y)| = 2$, $|L(v)| = 3$ for all $v \in V(C) - \{x,y\}$, and $|L(w)| = 5$ for all $w \in V(G) - V(C)$. Let $\tilde{G} = R(G)$ be a Type II reduction of G with respect to x, y. If $\tilde{G}$ is L-colorable, then G is L-colorable.

**Proof.** If $\tilde{G}$ was obtained from G by removing a separating 3-cycle with vertex set X for which X' is the vertex set of the connected component of G - X which contains neither x nor y and c is an L-coloring of G", then c may be extended to an L-coloring of G. Let L'(w) = L(w ) for all $w \in X$ and L (z) = {c(z)} for all $z \in X$, then $G[X \cup X']$ is L' -colorable by Theorem 4.7. Thus G is L-colorable. If $\tilde{G}$ was obtained from G by letting $\tilde{G} = R(G) = G_A$ where uv is a non-{x, y}-separating chord that splits G into two graphs $G_A$ and $G_B$ such that $G = G_A \cup G_B$,

$V(G_A)$ n $\cap$ $V(G_B) = \{u, v\}$, and $x, y \in V(G_A)$, and c is an L-coloring of $\tilde{G}$, then c may be extended to an L-coloring of G. Let L (w) = L(w) for all $w \in V(G_B) - \{u, v\}$, L (u) = {c(u)} and L (v) = {c(v)}, then $G_B$ is L -colorable by Theorem 1.13 . Thus, G is L-colorable.

**Theorem .** Let G = (V,E) be a plane graph and let C be the cycle that corresponds to the boundary of the unbounded face of G. Let $x,y \in V(C)$ be two nonadjacent vertices of C. Let $L : V \to 2^N$ be an arbitrary assignment of lists of colors to the vertices of G such that $|L(x)| = |L(y)| = 2$, $|L(v)| = 3$ for all $v \in V(C) - \{x, y\}$, and $|L(w)| = 5$ for al l $w \in V - V(C)$. Let $\tilde{G}$ be the Type II reduced graph of G with respect to x, y. If $\tilde{G}$ is L-colorable, then G is L-colorable.

This theorem follows from Lemma 4.14 because the Type II reduced graph of G is obtained from G via a series of Type II reductions.

## III. THEOREM

One of the main tools used in the following proof will be Theorem 4.9 and the notion of 3-extendability. A caterpillar is a tree in which all vertices of the graph are on or incident to a path which contains every vertex of degree at least two.

**Proof of Theorem .** Observe first that by Corollary , if the Type I I reduced graph of G is L-colorable, then G is L-colorable. Thus, assume that G is a Type II reduced graph with respect to x, y for the remainder of the proof.

1. Without loss of generality, assume G is a near-triangulation.

a. If $j \in \{2,k\}$, then the result follows from Theorem 1.13.

b. If $j \in \{3, k - 1\}$, assume j = 3. Add two adjacent vertices s and t with $s \sim \{x, x_2\}$ and $t \sim \{x_2, y\}$ so that s and t now lie on the cycle that corresponds to the unbounded face. Call this new graph G'. Let a and b be two colors not in any of the lists L. Assign to the vertices of G" the lists L' where L'(s) = {a}, L(t) = {b}, L' (x) = L(x) $\bigcup$ {a}, L $(x_2)$ = L$(x_2)$ $\bigcup$ {a, b}, L (y) = L(y) $\bigcup$ {b}, and L'(w) = L(w) for all $w \in V(G) - \{x, x_2, y\}$. By Theorem 1.13, G' is L-colorable and it follows that G is L-colorable.

c. If j {4, k - 2}, assume j = 4. Add a 3-path stu with $s \sim \{x, x_2\}$, $t \sim \{x_2, x_3\}$ and $u \sim \{x_3, y\}$ so that stu now lies on the cycle that corresponds to the unbounded face, see Figure 4.5. Call this new graph G" and let C" be the cycle that corresponds to the unbounded face. Let a,b and c be three colors not in any of the lists L. Assign to the vertices of G the lists L , where L(s) = {a}, L(t) = {b}, L(u) = {c}, L (x) = L(x) $\bigcup$ {a}, L $(x_2)$ = L$(x_2)$ $\bigcup$ {a, b}, L$(x_3)$ = L(x) $\bigcup$ {b, c}, L (y) = L(y) $\bigcup$ {c}, and L (w) = L(w) for all $w \in V(G) - \{x, x_2, x_3, y\}$. If G is 3-extendable with respect to stu, then that L -coloring of G provides an L-coloring of G. Thus, it remains to verify that G does not contain a subgraph H that is a generalized wheel with principal path stu and vertices of outercycle on C . Assume such a subgraph H exists in G . Observe that H cannot be a broken wheel because t does not have any neighbors with lists of size 3. Additionally, H cannot be a wheel because there is no vertex z in G such that $z \sim \{s, t, u\}$. Thus, H must be a generalized wheel formed by identifying principal edges of two wheels as seen in Figure 2.1c . However, this would require t to have degree 5 in H, a contradiction because t is of degree 4 in G . So by Theorem 4.9, G is 3-extendable with respect to stu.

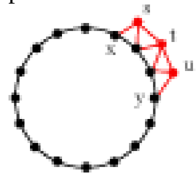



Figure 4.5: The addition of vertices $s, t, u$.

2. Let $G_T$ be the {x, y}-skeleton of G. Besides the fact that $G_T$ is a tree, some additional observations may be made. First, $G_T$ is indeed a caterpillar. There is also an underlying linear ordering of the chords of G. Consider the weak dual of G[C ]. This graph is a path $w_1$ w ….. $w_m$ whose endpoints correspond to the bounded faces of G that contain x and y, respectively. Let G , i = 1, . . ., m be the subgraph of G whose unbounded face has boundary that is the cycle corresponding to the vertex $w_i$ in the weak dual of G[C]. Additionally, each $w_i$ in the vertex set of the weak dual of G[C] corresponds to $P_1$ with vertices $u_{i-1}$, $u_i$, $u_{i+1}$ in $G_T$. Let $C_{P_i}$ be the coloring of $u_{i-1}u_i u_{i+1}$ with respect to $(G_i, L)$. As noted earlier, for each $P_i$, there is a $C_{P_i}$ for which the precoloring of $P_i$ does not extend to a proper L-coloring of $G_i$. Note that $x \in P_1$ and $y \in P_m$ uniquely. Say an {x, y}-skeleton has a "good" L-coloring if for all $G_i$, i = 1,. . . , m, the corresponding $P_i$ can all be simultaneously L-colored so that $P_i$ avoids $C_{P_i}$ .

**Claim .** If $G_T$ has a "good" L-coloring, then G is L-colorable.
The claim holds because the "good" L-coloring of $G_T$ may be extended to an L-coloring of G by Theorem 4.9 applied to each $G_i$.

**Claim .** $G_T$ has a "good" L-coloring.

**Proof of Claim** . By induction on m. If m = 1, then there are no chords and $x \sim y$, so the results follows by Theorem 1.13 .

So assume the result holds for m - 1 and consider $G_T$. Without loss of generality, assume $P_m = w_{m-1} w_m$ y and $C_{p_m} = (c_{m-1}, c_m, c_y)$. Let $G_T = G_T - \{y\}$. Let L $(w_m)$ = L$(w_m)$ - $\{c_m\}$ and L (w) = L(w) for all $w \in V(G_T) - \{w\}$. By induction, there is a "good" L -coloring c of $G_T$. This can be extended to a "good" L-coloring of $G_T$ by assigning to y a color from L(y) - $\{c(w_m)\}$.

By the above two claims, G is L-colorable.

## IV. CONCLUSION

We proved some theorems on {2, 2}-EXTENDABILITY OF PLANAR GRAPHS.
While every planar graph is 2-extendable, it is not the case that every planar graph is 3-extendable

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