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A Common Fixed Point Theorem in Dislocated Metric Space

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Abstract:- In this paper, we discuss the existence and unique of fixed point for two pairs of weakly compatible maps in dislocated metric space which generalizes and improves similar fixed point results.
Methods: Using familiar techniques, we extend the results in dislocated metric space.
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I. INTRODUCTION

Fixed point theory is one of the most dynamic research subjects in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. The most impressing result in this direction was given by Banach, called the Banach contraction mapping principle: Every contraction in a complete metric space has a unique fixed point. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. This celebrated result has been generalized in many abstract spaces for distinct operators.

In 2000, P. Hitzler and A. K. Seda [9] introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero.They also generalized the famous Banach contraction principle in this space. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. C.T. Aage and J. N. Salunke [2, 3], A. Isufati [1] established some important fixed point theorems in single and pair of mappings in dislocated metric space. K. Jha, D Panths[7] established a common fixed point theorem in dislocated metric spaces.

The purpose of this paper is to establish a common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric space.

II. PRELIMINARIES

For convenience we start with the followings definitions, lemmas and theorems.

Definition 2.1:[4] Let X be a non empty set and let d: $X \times X \to [0,\infty)$ be a function Satisfying the following conditions:

i. $d(x, y) = d(y, x)$
ii. $d(x, y)= d(y, x) = 0$ implies $x = y$.
iii. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called dislocated metric (or simply d-metric) on X.

Definition 2.2[9]: A sequence $\{X_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for $\in > 0$, there corresponds $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \in$

Definition 2.3[9]: A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0 \quad as \quad n \to \infty$.

In this case, x is called limit of $\{x_n\}$ (in d) and we write $x_n \to x$.

Definition 2.4[9]: A d-metric space (X, d) is called complete if every Cauchy sequence in it is convergent with respect to d.

Definition 2.5[9]: Let (X, d) be a d-metric space. A map T: X → X is called contraction if there exist a number λ with 0 ≤ λ < 1 such that $d(Tx,Ty) \le \lambda d(x,y)$
We state the following lemmas without proof.
Lemma 2.6: Let (X, d) be a d-metric space. If T: X → X is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in$ X.

Lemma 2.7 [9]: Limits in a d-metric space are unique.

Definition 2.8[5]: Let A and S be mappings from a metric space (X, d) into itself, then A and S are said to be weakly compatible if they commute at their coincident point; that is, Ax = Sx, for some x ∈ X implies ASx = SAx.

Theorem 2.9[8]: Let (X, d) be a complete d-metric space and let T: X → X be contraction mapping, and then T has a unique fixed point.

III. MAIN RESULT

Theorem 3.1: Let (X, d) be a complete d-metric space. Let A, B, S, T: X → X be continuous mappings satisfying,

i. T(X) ⊂ A(X), S(X) ⊂ B(X)
ii. The pairs (S, A) and (T, B) are weakly compatible and
iii. $d(Sx,Ty) \le \alpha d(Ax,Ty) + \beta d(Ax,By) + \gamma d(Ax,Sx) + \eta d(By,Ty) + \delta d(By,Sx)$

for all x, y ∈ X where $\alpha,\beta,\gamma,\eta,\delta \ge 0$ and $0 \le 2\alpha+\beta+\gamma+\eta+2\delta \prec 1.$ Then A, B, S, and T have a unique common fixed point.

Proof: Using condition (i) we define sequences $\{x_n\}$ *and* $\{y_n\}$ in X by the rule

$$y_{2n} = Bx_{2n+1} = Sx_{2n}$$

and $\quad y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}$, n = 0, 1 2. ……

If $y_{2n} = y_{2n+1}$ for some n then $Bx_{2n+2} = Tx_{2n+1}$

Therefore x_{2n+1} is coincidence point of B and T also if $y_{2n+1} = y_{2n+2}$ for some n then $Ax_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of S and A. Assume that $y_{2n} \ne y_{2n+1}$ for all n then we have

$$d(y_{2n},y_{2n+1}) = d(Sx_{2n},Tx_{2n+1})$$
$$\le \alpha d(Ax_{2n},Tx_{2n+1}) + \beta d(Ax_{2n},Bx_{2n+1}) + \gamma d(Ax_{2n},Sx_{2n}) + \eta d(Bx_{2n+1},Tx_{2n+1}) + \delta d(Bx_{2n+1},Sx_{2n})$$

$$\le \alpha d(y_{2n-1},y_{2n+1}) + \beta d(y_{2n-1},y_{2n}) + \gamma d(y_{2n-1},y_{2n}) + \eta d(y_{2n},y_{2n+1}) + \delta d(y_{2n},y_{2n})$$

$$\le \alpha d(y_{2n-1},y_{2n+1}) + \beta d(y_{2n-1},y_{2n}) + \gamma d(y_{2n-1},y_{2n}) + \eta d(y_{2n},y_{2n+1}) + \delta[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})]$$

$$\le (\alpha+\beta+\gamma+\delta)d(y_{2n-1},y_{2n}) + (\alpha+\eta+\delta)d(y_{2n},y_{2n+1})$$

$$d(y_{2n},y_{2n+1}) \le \frac{\alpha+\beta+\gamma+\delta}{1-\alpha-\eta-\delta}d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \le hd(y_{2n-1},y_{2n}) \qquad \text{where } h = \frac{\alpha+\beta+\gamma+\delta}{1-\alpha-\eta-\delta} \prec 1$$

This shows that

$$d(y_n,y_{n+1}) \le hd(y_{n-1},y_n) \le \ldots\ldots\ldots \le h^n d(y_0,y_1)$$

For any integer q > 0, we have

$$d(y_n, y_{n+q}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots\ldots + d(y_{n+q-1}, y_{n+q})$$
$$\leq (1 + h + h^2 + \ldots\ldots + h^n) d(y_n, y_{n+1})$$
$$\Rightarrow \quad d(y_n, y_{n+q}) \leq \frac{h^n}{1-h} d(y_0, y_1)$$

Since $0 \prec h \prec 1,\ h^n \to 0$ as $n \to \infty$, So we get $d(y_n, y_{n+q}) \to 0$. This implies that $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So there exist a point z in X such that $\{y_n\} \to z$ therefore the subsequences $\{Sx_{2n}\} \to z$, $\{Bx_{2n+1}\} \to z$, $\{Tx_{2n+1}\} \to z$, $\{Ax_{2n+2}\} \to z$.

Since T(X) ⊂ A(X) there exist a point u in X such that z = Au, so

$$d(Su, z) = d(Su, Tx_{2n+1})$$
$$\leq \alpha d(Au, Tx_{2n+1}) + \beta d(Au, Bx_{2n+1}) + \gamma d(Au, Su) + \eta d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Bx_{2n+1}, Su)$$
$$\leq \alpha d(z, Tx_{2n+1}) + \beta d(z, Bx_{2n+1}) + \gamma d(z, Su) + \eta d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Bx_{2n+1}, Su)$$

Now taking limit as n→ ∞ we get

$$d(Su, z) \leq \alpha d(z, z) + \beta d(z, z) + \gamma d(z, Su) + \eta d(z, z) + \delta d(z, Su)$$
$$\leq (\gamma + \delta) d(z, Su)$$

Which is a contradiction since $(\gamma + \delta) < 1$. So we have Su= Au= Z

Again since S(X) ⊂ B(X) there exist a point v in X such that z=Bv, We claim that z=Tv. If Z≠Tv then

$$d(z, Tv) = d(Su, Tv)$$
$$\leq \alpha d(Au, Tv) + \beta d(Au, Bv) + \gamma d(Au, Su) + \eta d(Bv, Tv) + \delta d(Bv, Su)$$
$$\leq \alpha d(z, Tv) + \beta d(z, z) + \gamma d(z, z) + \eta d(z, Tv) + \delta d(z, z)$$
$$\leq (\alpha + \eta) d(z, Tv)$$

A contradiction, since $(\alpha + \eta) < 1$ so we get $z = Tv$. Hence we claim that

$$Su = Au = Tv = Bv = z$$

Since the pair (S, A) are weakly compatible so by definition

$$SAu = ASu \quad \Rightarrow Sz = Az$$

Now we show that z is fixed point of S. If $Sz \neq z$ then

$$d(Sz, z) = d(Sz, Tv)$$
$$\leq \alpha d(Az, Tv) + \beta d(Az, Bv) + \gamma d(Az, Sz) + \eta d(Bv, Tv) + \delta d(Bv, Sz)$$
$$\leq \alpha d(Sz, z) + \beta d(Sz, z) + \gamma d(Sz, Sz) + \eta d(z, z) + \delta d(z, Sz)$$
$$\leq (\alpha + \beta + \delta) d(Sz, z)$$

This is a contradiction. So we have $Sz = z$. This implies that $Az = Sz = z$

Again the pair (T, B) are weakly compactible so by definition $TBv = BTv \Rightarrow Tz = Bz$

Now we show that z is fixed point of T. If $Tz \neq z$ then

$$d(z, Tz) = d(Sz, Tz)$$
$$\leq \alpha d(Az, Tz) + \beta d(Az, Bz) + \gamma d(Az, Sz) + \eta d(Bz, Tz) + \delta d(Bz, Sz)$$
$$\leq \alpha d(z, Tz) + \beta d(z, Tz) + \gamma [d(Az, Tz) + d(Tz, Sz)] + \eta d(Tz, Tz) + \delta d(Tz, Sz)$$
$$\leq (\alpha + \beta + 2\gamma + \delta) d(z, Tz)$$

This is a contradiction. This implies that $z = Tz$ hence we have $Az = Bz = Sz = Tz = z$. This shows that z is common fixed point of self mapping A, B, S and T.

Uniqueness: Let $u \neq v$ be two common fixed points of the mappings A, B, S & T then we have

$$d(u, v) = d(Su, Tv)$$
$$\leq \alpha d(u, v) + \beta d(u, v) + \gamma d(u, u) + \eta d(v, v) + \delta d(v, u)$$
$$\leq (\alpha + \beta + \gamma + \delta) d(u, v)$$

A contradiction this shows that d(u, v) = 0. Since (X, d) is a dislocated metric space. So we have u = v
This establishes the theorem.
Example: Let X =[0, 1] and d be a usual metric space. Let the mappings A, B, S and T be defined by
$Sx = 0,\quad Ax = \frac{x}{2} \quad Tx = \frac{x}{6} \quad \& \quad Bx = x$ then for $\alpha = \frac{1}{8}$, $\beta = \frac{1}{7}$, $\gamma = \frac{1}{8}$ $\eta = \frac{1}{9}$ and $\delta = \frac{1}{9}$ the mappings A ,B ,S and T satisfies all the conditions of above theorem (3.1)and So x = 0 is unique common fixed point of the four mappings A, B, S and T.
Corollary 3.2: Let (X, d) be a complete d-metric space. Let S, T: X → X be continuous mappings satisfying,

$$d(Sx, Ty) \le \alpha d(x, Ty) + \beta d(x, y) + \gamma d(x, Sx) + \eta d(y, Ty) + \delta d(y, Sx),$$ for all x, y ∈ X,

where α, β, γ, η, δ ≥ 0, 0 ≤ 2α + β +γ+η+2δ <1, Then S and T have a unique common fixed point.
Proof: If we take A = B = I an identity mapping in above theorem 3.1 and Follow the similar proof as that in theorem, we can establish this corollary 3.2.
If we take S = T then the corollary 3.2 is reduced to
Corollary 3.3: Let (X, d) be a complete d-metric space. Let T: X → X bea continuous mapping satisfying,

$$d(Tx, Ty) \le \alpha d(x, Ty) + \beta d(x, y) + \gamma d(x, Tx) + \eta d(y, Ty) + \delta d(y, Tx),$$ for all x, y ∈ X,

where α, β, γ,η,δ ≥ 0, 0 ≤ 2α + β +γ+η+2δ <1. Then T has a unique fixed point.
Corollary 3.4: Let (X, d) be a complete d-metric space. Let S, T : X → X be continuous mappings satisfying,

$$d(Sx, Ty) \le \alpha d(Tx, Ty) + \beta d(Tx, Sy) + \gamma d(Tx, Sx) + \eta d(Sy, Ty) + \delta d(Sy, Sx),$$

For all x, y ∈ X, whereα, β, γ, η, δ ≥ 0 and 0≤ 2α+ β +γ+η+2δ <1. Then S and T have aunique common fixed point.
Proof: If we take A = T and B = S in Theorem 3.1 and apply the similar proof, we can establish this corollary 3.4.

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