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Five-Dimensional Finsler Spaces with T-Tensor of Some Special forms

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ABSTRACT: The T−tensor played an important role in the Finsler geometry. In this paper, we discuss a five-dimensional Finsler space whose T−tensor is of special forms.
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I. INTRODUCTION

Let M^5 be a five-dimensional smooth manifold and $F^5 = (M^5, L)$ be a five-dimensional Finsler space equipped with a metric function $L(x, y)$ on M^5. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$$l_i = \dot{\partial}_i L, \quad g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2, \quad h_{ij} = L\dot{\partial}_i \dot{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$$

respectively.
The torsion vector C^i is defined by $C^i = C^i_{jk} g^{jk}$.

Throughout the paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial / \partial y^i$ and $\partial / \partial x^i$ respectively. The Cartan connection in the Finsler space is given as $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. The *h*− and *v*−covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G^h_j - F^r_{ij} X_r, \tag{1.1}$$

and

$$X_i |_j = \dot{\partial}_j X_i - C^r_{ij} X_r. \tag{1.2}$$

In 1972, H. Kawaguchi [1] and M. Matsumoto [2] independently found an important tensor

$$T_{hijk} = LC_{hij}|_k + C_{hij} l_k + C_{hik} l_j + C_{hkj} l_i + C_{kij} l_h. \tag{1.3}$$

This is called the *T*−tensor. It is completely symmetric in its indices. The vanishing of *T*−tensor is called *T*−condition.

U. P. Singh et al. [3, 4] studied three-dimensional Finsler spaces with *T*−tensor of the following forms:

(A) $T_{hijk} = \rho(h_{hi} h_{jk} + h_{hj} h_{ik} + h_{hk} h_{ij})$,

(B) $T_{hijk} = h_{hi} P_{jk} + h_{hj} P_{ik} + h_{hk} P_{ij} + h_{ij} P_{hk} + h_{ik} P_{hj} + h_{jk} P_{hi}$,

(C) $T_{hijk} = \rho C_h C_i C_j C_k + a_h C_i C_j C_k + a_i C_h C_j C_k + a_j C_h C_i C_k + a_k C_h C_i C_j$,

where P_{ij} are the components of a tensor field, a_h are the components of a covariant vector field and ρ is a scalar. Present authors studied the theory of five-dimensional Finsler space. In this paper [9-11], we discuss five-dimensional Finsler spaces with *T*−tensor of such forms.

II. FIVE-DIMENSIONAL FINSLER SPACE

The Miron frame for a five-dimensional Finsler space is constructed by the unit vectors $(e^i_{1)}, e^i_{2)}, e^i_{3)}, e^i_{4)}, e^i_{5)})$. The first vector $e^i_{1)}$ is the normalized supporting element l^i and the second $e^i_{2)}$, is the normalized torsion vector $m^i = C^i / C$, the third $e^i_{3)} = n^i$, the fourth $e^i_{4)} = p^i$ and the fifth $e^i_{5)} = q^i$ are constructed by $g_{ij} e^i_{\alpha)} e^j_{\beta)} = \delta_{\alpha\beta}$. We suppose that the length *C* of the vector C^i does not vanish, i.e., the space is non-Riemannian. With respect to this frame, the scalar components of an arbitrary tensor T^i_j are defined by

$$T_{\alpha\beta} = T^i_j e_{\alpha)i}\, e^j_{\beta)}, \tag{2.1}$$

from which, we get

$$T^i_j = T_{\alpha\beta} e^i_{\alpha)}\, e_{\beta)j}, \tag{2.2}$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$.

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}| L$ be scalar components of the *h*− and *v*−covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ respectively of the vectors $e^i_{\alpha)}$, then

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}, \tag{2.3}$$

and

$$L e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma} e^i_{\beta)}\, e_{\gamma)j}. \tag{2.4}$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called *h*− and *v*−connection scalars respectively and are positively homogeneous of degree zero in *y*. Orthogonality of the Miron frame yields [5] $H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}$ and $V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$. Also, we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{|\beta}\delta_{|\gamma}$.

Now, we define Finsler vector fields:

$$h_i = H_{2)3\beta}\, e_{\beta)i}, \qquad J_i = H_{2)4\beta} e_{\beta)i}, \qquad k_i = H_{2)5\beta} e_{\beta)i},$$
$$h_i' = H_{3)4\beta}\, e_{\beta)i}, \qquad J_i' = H_{3)5\beta} e_{\beta)i}, \qquad k_i' = H_{4)5\beta} e_{\beta)i},$$

and

$$u_i = V_{2)3\beta}\, e_{\beta)i}, \qquad v_i = V_{2)4\beta} e_{\beta)i}, \qquad w_i = V_{2)5\beta} e_{\beta)i},$$
$$u_i' = V_{3)4\beta}\, e_{\beta)i}, \qquad v_i' = V_{3)5\beta} e_{\beta)i}, \qquad w_i' = V_{4)5\beta} e_{\beta)i},$$

Definition. The Finsler vector fields $(h_i, J_i, k_i, h_i', J_i', k_i')$ are called *h*−connection vectors and the vector fields $(u_i, v_i, w_i, u_i', v_i', w_i')$ are called *v*−connection vectors.

The scalars $H_{2)3\beta}$, $H_{2)4\beta}$, $H_{2)5\beta}$, $H_{3)4\beta}$, $H_{3)5\beta}$, $H_{4)5\beta}$ and $V_{2)3\beta}$, $V_{2)4\beta}$, $V_{2)5\beta}$, $V_{3)4\beta}$, $V_{3)5\beta}$, $V_{4)5\beta}$ are considered as the scalar components h_β, J_β, k_β, h_β', J_β', k_β' and u_β, v_β, w_β, u_β', v_β', w_β' of the *h*− and *v*−connection vectors respectively with respect to the orthonormal frame.

From (2.4), we get

$$\begin{aligned}
&\text{(a)}\quad L e^i_{1)}|_j = L l^i|_j = m^i m_j + n^i n_j + p^i p_j + q^i q_j = h^i_j,\\
&\text{(b)}\quad L e^i_{2)}|_j = L m^i|_j = -l^i m_j + n^i u_j + p^i v_j + q^i w_j,\\
&\text{(c)}\quad L e^i_{3)}|_j = L n^i|_j = -l^i n_j - m^i u_j + p^i u_j' + q^i v_j',\\
&\text{(d)}\quad L e^i_{4)}|_j = L p^i|_j = -l^i p_j - m^i v_j - n^i u_j' + q^i w_j',\\
&\text{(e)}\quad L e^i_{5)}|_j = L q^i|_j = -l^i q_j - m^i w_j - n^i v_j' - p^i w_j'.
\end{aligned} \tag{2.5}$$

Since m_i, n_i, p_i, q_i are homogeneous function of degree zero in y^i, we have

$$L m^i|_j\, l^j = L n^i|_j\, l^j = L p^i|_j\, l^j = L q^i|\, j l^j = 0.$$

These imply $u_1 = v_1 = w_1 = u_1' = v_1' = w_1' = 0$. Consequently, we have

Proposition 2.1. The first scalar components u_1, v_1, w_1, u_1', v_1', w_1' of *v*−connection vectors u_i, v_i, w_i, u_i', v_i', w_i' vanish identically.

Let $C_{\alpha\beta\gamma}$ be the scalar components of LC_{ijk} with respect to the Miron frame, i.e.,

$$LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha)i} e_{\beta)j} e_{\gamma)k}. \tag{2.6}$$

The main scalars of a five-dimensional Finsler space are given by [9-10]

$$\begin{array}{lllll} C_{222} = H, & C_{233} = I, & C_{244} = K, & C_{255} = M, & C_{333} = J, \\ C_{344} = J', & C_{444} = H', & C_{334} = I', & C_{234} = K', & C_{355} = J'', \\ C_{455} = M', & C_{555} = H'', & C_{335} = I'', & C_{445} = K'', & C_{235} = N, \\ C_{245} = N', & C_{345} = M'', & & & \end{array}$$

we have

$$C_{322} = -(J + J' + J''), \quad C_{224} = -(H' + I' + M'), \quad C_{225} = -(H'' + I'' + M'')$$

and

$$H + I + K + M = LC \tag{2.7}$$

The scalar components $T_{\alpha\beta;\gamma}$ of $LT^i_j|_k$ are written in the form [5]

$$T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} V_{\mu\alpha\gamma} + T_{\alpha\mu} V_{\mu\beta\gamma}. \tag{2.8}$$

The explicit form of $C_{\alpha\beta\gamma;\delta}$ is obtained as follows:

$$\begin{aligned}
C_{222;\delta} &= H_{;\delta} + 3(J + J' + J''u_\delta + 3(H' + I' + M')v_\delta + 3(H'' + I'' + K'')w_\delta, \\
C_{223;\delta} &= -(J + J' + J'')_{;\delta} + (H - 2I)u_\delta - 2K'v_\delta - 2Nw_\delta + (H' + I' + M')u_\delta' \\
&\quad + (H'' + I'' + M'')v_\delta', \\
C_{224;\delta} &= -(H' + I' + M')_{;\delta} - 2K'u_\delta + (H - 2K)v_\delta - 2N'w_\delta - (J + J' + J'')u_\delta' \\
&\quad + (H'' + I'' + K'')w_\delta', \\
C_{225;\delta} &= -(H'' + I'' + K'')_{;\delta} - 2Nu_\delta - 2N'v_\delta + (H - 2M)w_\delta - (J + J' + J'')v_\delta' \\
&\quad - (H' + I' + M')w_\delta', \\
C_{233;\delta} &= I_{;\delta} - (3J + 2J' + 2J'')u_\delta - I'v_\delta - I''w_\delta - 2Nv_\delta' - 2K'u_\delta', \\
C_{234;\delta} &= K_{;\delta}' - (2I' + H' + M')u_\delta - (2J' + J + J'')v_\delta - M''w_\delta - (K - I)u_\delta' \\
&\quad - N'v_\delta' - Nw_\delta', \\
C_{235;\delta} &= N_{;\delta} - (2I'' + H'' + K'')u_\delta - M''v_\delta - (J + J' + 2J'')w_\delta \\
&\quad - N'u_\delta' - (M - I)v_\delta' + K'w_\delta', \\
C_{244;\delta} &= K_{;\delta} - J'u_\delta - (3H' + 2I' + 2M')v_\delta + 2Ku_\delta' - K''w_\delta - 2N'w_\delta', \\
C_{245;\delta} &= N_{;\delta}' - M''u_\delta - (H'' + I'' + 2K'')v_\delta + Nu_\delta' - (H' + I' + 2M')w_\delta \\
&\quad + K'v_\delta' + (K - M)w_\delta', \\
C_{255,\delta} &= M_{;\delta} - J''u_\delta - M'v_\delta - (3H'' + 2I'' + 2K'')w_\delta + 2Nv_\delta' + 2N'w_\delta', \\
C_{333,\delta} &= J_{;\delta} + 3(Iu_\delta - I'u_\delta' - I''v_\delta'), \\
C_{334;\delta} &= I_{;\delta}' + 2K'u_\delta + Iv_\delta + (J - 2J')u_\delta' - 2M''v_\delta' - I''w_\delta', \\
C_{335;\delta} &= I_{;\delta}'' + 2Nu_\delta - 2M''u_\delta' + (J - 2J'')v_\delta' + Iw_\delta + I'w_\delta', \\
C_{344;\delta} &= J_{;\delta}' + Ku_\delta + 2K'v_\delta - (H - 2I')u_\delta' - K''v_\delta' - 2M''w_\delta', \\
C_{345;\delta} &= M_{;\delta}'' + N'u_\delta + Nv_\delta + (I'' - K'')u_\delta' + K'w_\delta + (I' - M')v_\delta' \\
&\quad + (J' - J'')w_\delta',
\end{aligned} \tag{2.9}$$

$$C_{355;\delta} = J_{;\delta}'' + Mu_\delta - M'u_\delta' + 2Nw_\delta - (H'' - 2I'')v_\delta' + 2M''w_\delta',$$
$$C_{444;\delta} = H_{;\delta}' + 3(Kv_\delta + J'u_\delta' - K''w_\delta'),$$
$$C_{445;\delta} = K_{;\delta}'' + 2N'v_\delta + 2M''u_\delta' + Kw_\delta + J'v_\delta' + (H' - 2M')w_\delta',$$
$$C_{455;\delta} = M_{;\delta}' + Mv_\delta + J''u_\delta' + 2N'w_\delta + 2M'v_\delta' - (H'' - 2K'')w_\delta',$$
$$C_{555;\delta} = H_{;\delta}'' + 3(Mw_\delta + J''v_\delta' + M'w_\delta').$$
$$C_{1\beta\gamma;\delta} = -C_{\beta\gamma\delta},$$

where $H_{;\delta} = L(\dot{\partial}_k H)\, e^k_{\delta)}$. From (2.7) and (2.9), we get

$$\begin{aligned}
C_{222;\delta} + C_{233;\delta} + C_{244;\delta} + C_{255;\delta} &= H_{;\delta} + I_{;\delta} + K_{;\delta} + M_{;\delta} \\
&= (H + I + K + M)_{;\delta} = (LC)_{;\delta} \\
C_{322;\delta} + C_{333;\delta} + C_{344;\delta} + C_{355;\delta} &= LCu_\delta, \\
C_{224;\delta} + C_{334;\delta} + C_{444;\delta} + C_{455;\delta} &= LCv_\delta, \\
C_{225;\delta} + C_{335;\delta} + C_{445;\delta} + C_{555;\delta} &= LCw_\delta.
\end{aligned} \tag{2.10}$$

From (2.6), it follows that

$$L^2 C_{ijk}|_h + LC_{ijk} l_h = C_{\alpha\beta\gamma;\delta} e_{\alpha)i} e_{\beta)j} e_{\gamma)k} e_{\delta)h},$$

which implies

$$L^2 C_{ijk}\,|_h = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta})\, e_{\alpha)i} e_{\beta)j} e_{\gamma)k} e_{\delta)h}. \tag{2.11}$$

From (1.3) and (2.11), we get

$$LT_{hijk} = (C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma})\, e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}. \tag{2.12}$$

Since the tensor $C_{hij}|_k$ is completely symmetric in its indices, from (2.11), we get

$$C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\delta;\gamma} = C_{\alpha\beta\gamma}\delta_{1\delta} - C_{\alpha\beta\delta}\delta_{1\gamma}. \tag{2.13}$$

In view of (2.13), equation (2.10) gives

$$\begin{aligned}
LCu_2 &= C_{322;2} + C_{333;2} + C_{344;2} + C_{355;2} = C_{222;3} + C_{233;3} + C_{244;3} + C_{255;3} = (LC)_{;3}, \\
LCv_2 &= C_{224;2} + C_{334;2} + C_{444;2} + C_{455;2} = C_{222;4} + C_{233;4} + C_{244;4} + C_{255;4} = (LC)_{;4}, \\
LCu_4 &= C_{322;4} + C_{333;4} + C_{344;4} + C_{355;4} = C_{224;3} + C_{334;3} + C_{444;3} + C_{455;3} = (LC)v_3, \\
LCu_5 &= C_{322;5} + C_{333;5} + C_{344;5} + C_{355;5} = C_{225;3} + C_{335;3} + C_{445;3} + C_{555;3} = (LC)w_3, \\
LCv_5 &= C_{224;5} + C_{334;5} + C_{444;5} + C_{455;5} = C_{225;4} + C_{335;4} + C_{445;4} + C_{555;4} = (LC)w_4, \\
LCw_2 &= C_{225;2} + C_{335;2} + C_{445;2} + C_{555;2} = C_{222;5} + C_{233;5} + C_{244;5} + C_{255;5} = (LC)_{;5}.
\end{aligned} \tag{2.14}$$

Since $L_{;3} = L(\dot{\partial}_i L)\, e^i_{3)} = Ll_i n^i = 0$, $L_{;4} = L(\dot{\partial}_i L)\, e^i_{4)} = Ll_i p^i = 0$ and $L_{;5} = L(\dot{\partial}_i L)\, e^i_{5)} = Ll_i q^i = 0$, we have

Proposition 2.2. The scalar components u_2, v_2 and w_2 of the v−connection vectors u_i, v_i and w_i of a five-dimensional Finsler space are given by

$$u_2 = C^{-1}C_{;3}, \qquad v_2 = C^{-1}C_{;4}, \qquad w_2 = C^{-1}C_{;5},$$

and the scalar components u_4, u_5, v_3, v_5, w_3 and w_4 are related by

$$u_4 = v_3, \qquad u_5 = w_3, \qquad v_5 = w_4.$$

III. T–TENSOR OF FORM (A)

A Finsler space is C−reducible if and only if the T−tensor is of the form (A) for $\rho \neq 0$[6–7]. Let F^5 be a five-dimensional Finsler space with T−tensor of the form (A). The scalar components of the angular metric tensor h_{ij} are given by

$$h_{ij} = (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\, e_{\alpha)i} e_{\beta)j},$$

therefore in view of (2.12) and (A), we have

$$(C_{\alpha\beta\gamma;\delta}+C_{\beta\gamma\delta}\delta_{1\alpha}+C_{\alpha\gamma\delta}\delta_{1\beta}+C_{\alpha\beta\delta}\delta_{1\gamma})=\rho L\{(\delta_{\alpha\beta}-\delta_{1\alpha}\delta_{1\beta})(\delta_{\gamma\delta}-\delta_{1\gamma}\delta_{1\delta})+(\delta_{\alpha\gamma}-\delta_{1\alpha}\delta_{1\gamma})(\delta_{\beta\delta}-\delta_{1\beta}\delta_{1\delta})+(\delta_{\alpha\delta}-\delta_{1\alpha}\delta_{1\delta})(\delta_{\beta\gamma}-\delta_{1\beta}\delta_{1\gamma})\},$$

which gives

$$\begin{aligned}
&C_{222;\delta}=3\rho L\delta_{2\delta}, && C_{233;\delta}=\rho L\delta_{2\delta}, && C_{244;\delta}=\rho L\delta_{2\delta}, && C_{255;\delta}=\rho L\delta_{2\delta},\\
&C_{322;\delta}=\rho L\delta_{3\delta}, && C_{333;\delta}=3\rho L\delta_{3\delta}, && C_{344;\delta}=\rho L\delta_{3\delta}, && C_{355;\delta}=\rho L\delta_{3\delta},\\
&C_{224;\delta}=\rho L\delta_{4\delta}, && C_{225;\delta}=\rho L\delta_{5\delta}, && C_{444;\delta}=3\rho L\delta_{4\delta}, && C_{334;\delta}=\rho L\delta_{4\delta},\\
&C_{335;\delta}=\rho L\delta_{5\delta}, && C_{455;\delta}=\rho L\delta_{4\delta}, && C_{555;\delta}=3\rho L\delta_{5\delta}, && C_{445;\delta}=\rho L\delta_{5\delta}.
\end{aligned} \tag{3.1}$$

Putting (3.1) into (2.10), we get

$$(LC)_{;\delta}=6\rho L\delta_{2\delta},\quad LCu_\delta=6\rho L\delta_{3\delta},\quad LCv_\delta=6\rho L\delta_{4\delta},\quad LCw_\delta=6\rho L\delta_{5\delta}.$$

Again from the first equation of (3.1), we get

$$C_{222;\delta}=H_{;\delta}+3(J+J'+J'')u_\delta+3(H'+I'+M')v_\delta+3(H''+I''+K'')w_\delta=3\rho L\delta_{2\delta}.$$

Thus, we have

Theorem 3.1. If the T−tensor of a five-dimensional Finsler space is of the form (A), then ρ is given by

$$\rho=\frac{H_{;2}}{3L}=\frac{1}{6}C_{;2}=\frac{1}{6}Cu_3=\frac{1}{6}Cv_4=\frac{1}{6}Cw_5.$$

Theorem 3.2. The scalar components of v−connection vectors u_i and v_i of a five-dimensional Finsler space with T−tensor of the form (A) are given by

$$\begin{aligned}
&u_1=0, && u_2=0, && u_3=C^{-1}C_{;2}, && u_4=0, && u_5=0,\\
&v_1=0, && v_2=0, && v_3=0, && v_4=C^{-1}C_{;2}, && v_5=0,\\
&w_1=0, && w_2=0, && w_3=0, && w_4=0, && w_5=C^{-1}C_{;2}.
\end{aligned}$$

IV. T–TENSOR OF FORM (B)

Ikeda [8] showed that for an n−dimensional Finsler space with T−tensor of the form (B)

$$T_{hijk}=h_{hi}P_{jk}+h_{hj}P_{ik}+h_{hk}P_{ij}+h_{ij}P_{hk}+h_{ik}P_{hj}+h_{jk}P_{hi},$$

we get

$$P_{ij}=\frac{1}{n+3}\Big\{T_{ij}-\frac{T}{2(n+1)}h_{ij}\Big\},$$

where $T_{ij}=T_{hijk}g^{hk}$ and $T=T_{ij}g^{ij}$.

Therefore (B) becomes

$$T_{hijk}=\frac{1}{n+3}(h_{hi}T_{jk}+h_{hj}T_{ik}+h_{hk}T_{ij}+h_{ij}T_{hk}+h_{ik}T_{hj}+h_{jk}T_{hi})-\frac{T}{(n+1)(n+3)}(h_{hi}h_{jk}+h_{hj}h_{ik}+h_{hk}h_{ij}).$$

Thus, for a five-dimensional Finsler space, we have

$$T_{hijk}=\frac{1}{8}[(h_{hi}T_{jk}+h_{hj}T_{ik}+h_{hk}T_{ij}+h_{ij}T_{hk}+h_{ik}T_{hj}+h_{jk}T_{hi})-\frac{T}{6}(h_{hi}h_{jk}+h_{hj}h_{ik}+h_{hk}h_{ij})]. \tag{4.1}$$

Let $T_{\alpha\beta}$ be the scalar components of LT_{hi}, i.e.,

$$LT_{hi}=T_{\alpha\beta}e_{\alpha)h}\,e_{\beta)i}.$$

In view of (2.12) and (4.1), we get

$$(C_{\alpha\beta\gamma;\delta}+C_{\beta\gamma\delta}\delta_{1\alpha}+C_{\alpha\gamma\delta}\delta_{1\beta}+C_{\alpha\beta\delta}\delta_{1\gamma})=\frac{1}{8}[\{(\delta_{\alpha\beta}-\delta_{1\alpha}\delta_{1\beta})T_{\gamma\delta}+(\delta_{\alpha\gamma}-\delta_{1\alpha}\delta_{1\gamma})T_{\beta\delta}$$
$$+(\delta_{\alpha\delta}-\delta_{1\alpha}\delta_{1\delta})T_{\beta\gamma}+(\delta_{\beta\gamma}-\delta_{1\beta}\delta_{1\gamma})T_{\alpha\delta}+(\delta_{\beta\delta}-\delta_{1\beta}\delta_{1\delta})T_{\alpha\gamma}+(\delta_{\gamma\delta}-\delta_{1\gamma}\delta_{1\delta})T_{\alpha\beta}\}$$
$$-\frac{LT}{6}[\{(\delta_{\alpha\beta}-\delta_{1\alpha}\delta_{1\beta})(\delta_{\gamma\delta}-\delta_{1\gamma}\delta_{1\delta})+(\delta_{\alpha\gamma}-\delta_{1\alpha}\delta_{1\gamma})(\delta_{\beta\delta}-\delta_{1\beta}\delta_{1\delta})$$
$$+(\delta_{\alpha\delta}-\delta_{1\alpha}\delta_{1\delta})(\delta_{\beta\gamma}-\delta_{1\beta}\delta_{1\gamma})\},$$

which gives

$$\begin{aligned}
C_{222;\delta}&=\frac{1}{8}[3T_{2\delta}+3T_{22}\delta_{2\delta}-\frac{1}{2}LT\delta_{2\delta}],\\
C_{233;\delta}&=\frac{1}{8}[T_{33}\delta_{2\delta}+T_{2\delta}+2T_{23}\delta_{3\delta}-\frac{1}{6}LT\delta_{2\delta}],\\
C_{244;\delta}&=\frac{1}{8}[T_{44}\delta_{2\delta}+T_{2\delta}+2T_{24}\delta_{4\delta}-\frac{1}{6}LT\delta_{2\delta}],\\
C_{255;\delta}&=\frac{1}{8}[T_{55}\delta_{2\delta}+T_{2\delta}+2T_{25}\delta_{5\delta}-\frac{1}{6}LT\delta_{2\delta}],\\
C_{322;\delta}&=\frac{1}{8}[T_{22}\delta_{3\delta}+T_{3\delta}+2T_{33}\delta_{2\delta}-\frac{1}{6}LT\delta_{3\delta}],\\
C_{333;\delta}&=\frac{1}{8}[3T_{3\delta}+3T_{33}\delta_{3\delta}-\frac{1}{2}LT\delta_{3\delta}],\\
C_{344;\delta}&=\frac{1}{8}[T_{44}\delta_{3\delta}+T_{3\delta}+2T_{34}\delta_{4\delta}-\frac{1}{6}LT\delta_{3\delta}],\\
C_{355;\delta}&=\frac{1}{8}[T_{55}\delta_{3\delta}+T_{3\delta}+2T_{35}\delta_{5\delta}-\frac{1}{6}LT\delta_{3\delta}],\\
C_{224;\delta}&=\frac{1}{8}[T_{22}\delta_{4\delta}+T_{4\delta}+2T_{24}\delta_{2\delta}-\frac{1}{6}LT\delta_{4\delta}],\\
C_{225;\delta}&=\frac{1}{8}[T_{22}\delta_{5\delta}+T_{5\delta}+2T_{25}\delta_{2\delta}-\frac{1}{6}LT\delta_{5\delta}],\\
C_{444;\delta}&=\frac{1}{8}[3T_{4\delta}+3T_{44}\delta_{4\delta}-\frac{1}{2}LT\delta_{4\delta}],\\
C_{334;\delta}&=\frac{1}{8}[T_{4\delta}+2T_{34}\delta_{3\delta}+T_{33}\delta_{4\delta}-\frac{1}{6}LT\delta_{4\delta}],\\
C_{335;\delta}&=\frac{1}{8}[T_{5\delta}+2T_{35}\delta_{3\delta}+T_{33}\delta_{5\delta}-\frac{1}{6}LT\delta_{5\delta}],\\
C_{455;\delta}&=\frac{1}{8}[T_{4\delta}+2T_{45}\delta_{5\delta}+T_{55}\delta_{4\delta}-\frac{1}{6}LT\delta_{4\delta}],\\
C_{555;\delta}&=\frac{1}{8}[3T_{5\delta}+3T_{55}\delta_{5\delta}-\frac{1}{2}LT\delta_{5\delta}],\\
C_{445;\delta}&=\frac{1}{8}[T_{5\delta}+2T_{45}\delta_{4\delta}+T_{44}\delta_{5\delta}-\frac{1}{6}LT\delta_{5\delta}].
\end{aligned}\tag{4.2}$$

Putting (4.2) into (2.10), we get

$$(LC)_{;\delta} = \frac{1}{8}[6T_{2\delta} + (3T_{22} + T_{33} + T_{44} + T_{55})\delta_{2\delta} + 2T_{23}\delta_{3\delta} + 2T_{24}\delta_{4\delta} + 2T_{25}\delta_{5\delta} - LT\delta_{2\delta}],$$

$$LCu_{\delta} = \frac{1}{8}[6T_{3\delta} + (3T_{33} + T_{22} + T_{44} + T_{55})\delta_{3\delta} + 2T_{23}\delta_{2\delta} + 2T_{34}\delta_{4\delta} + 2T_{35}\delta_{5\delta} - LT\delta_{3\delta}],$$

$$LCv_{\delta} = \frac{1}{8}[6T_{4\delta} + (3T_{44} + T_{22} + T_{33} + T_{55})\delta_{4\delta} + 2T_{24}\delta_{2\delta} + 2T_{34}\delta_{3\delta} + 2T_{45}\delta_{5\delta} - LT\delta_{4\delta}],$$

$$LCw_{\delta} = \frac{1}{8}[6T_{5\delta} + (T_{22} + T_{33} + T_{44} + 3T_{55})\delta_{5\delta} + 2T_{25}\delta_{2\delta} + 2T_{35}\delta_{3\delta} + 2T_{45}\delta_{4\delta} - LT\delta_{5\delta}].$$

Therefore,

$$\begin{aligned}
&(LC)_{;2} = \frac{1}{8}\{9T_{22} + T_{33} + T_{44} + T_{55} - LT\},\\
&(LC)_{;3} = T_{23}, \qquad (LC)_{;4} = T_{24}, \qquad (LC)_{;5} = T_{25},\\
&LCu_2 = T_{23}, \qquad LCu_3 = \frac{1}{8}\{T_{22} + 9T_{33} + T_{44} + T_{55} - LT\},\\
&LCu_4 = T_{34}, \qquad LCu_5 = T_{35}, \qquad LCv_2 = T_{24}, \qquad LCv_3 = T_{34},\\
&LCv_4 = \frac{1}{8}\{T_{22} + T_{33} + 9T_{44} + T_{55} - LT\}, \qquad LCv_5 = T_{45},\\
&LCw_2 = T_{25}, \qquad LCw_3 = T_{35}, \qquad LCw_4 = T_{45},\\
&LCw_5 = \frac{1}{8}\{T_{22} + T_{33} + T_{44} + 9T_{55} - LT\}.
\end{aligned} \tag{4.3}$$

From $T = T_{ij}\, g^{ij}$, we find

$$LT = T_{\alpha\beta}\delta_{\alpha\beta} = T_{\alpha\alpha} = T_{22} + T_{33} + T_{44} + T_{55}.$$

Thus, in view of (4.3), we have

Theorem 4.1. If the *T*−tensor of a five-dimensional Finsler space is of the form (B), the scalar components of the tensor T_{ij} are given by

$$\begin{aligned}
&T_{1\alpha} = 0, \qquad T_{22} = (LC)_{;2}, \qquad T_{33} = LCu_3,\\
&T_{44} = LCv_4, \qquad T_{55} = LCw_5, \qquad T_{23} = LCu_2 = (LC)_{;3},\\
&T_{24} = LCv_2 = (LC)_{;4}, \quad T_{25} = LCw_2 = (LC)_{;5}, \quad T_{34} = LCu_4 = LCv_3,\\
&T_{35} = LCu_5 = LCw_3, \quad T_{45} = LCw_4 = LCv_5,
\end{aligned}$$

and $\quad T = C_{;2} + Cu_3 + Cv_4 + Cw_5$.

V. T–TENSOR OF FORM (C)

U. P. Singh et al. [4] showed that the *T*−tensor of a *C*−2 like Finsler space is of the form (C)

$$T_{hijk} = \rho C_h C_i C_j C_k + a_h C_i C_j C_k + a_i C_h C_j C_k + a_j C_h C_i C_k + a_k C_h C_i C_j .$$

Let a_α be the scalar components of La_i , i.e.,

$$La_i = a_\alpha\, e_{\alpha)i} .$$

Since $e_{2)i} = C_i / C$, we get $C_i = C\delta_{2\alpha}\, e_{\alpha)i}$.

Therefore in view of (2.12) and (C), we have

$$\begin{aligned}(C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma}) = \rho LC^4\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta} + C^3(a_\alpha\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}\\
+ a_\beta\delta_{2\alpha}\delta_{2\gamma}\delta_{2\delta} + a_\gamma\delta_{2\alpha}\delta_{2\beta}\delta_{2\delta} + a_\delta\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}),\end{aligned}$$

which gives

$$\begin{array}{llll} C_{222;\delta} = C^3(\rho LC + 3a_2)\delta_{2\delta} + C^3 a_\delta, & C_{233;\delta} = 0, & C_{244;\delta} = 0, & C_{255;\delta} = 0, \\ C_{322;\delta} = C^3 a_3 \delta_{2\delta}, & C_{333;\delta} = 0, & C_{344;\delta} = 0, & C_{355;\delta} = 0, \\ C_{224;\delta} = C^3 a_4 \delta_{2\delta}, & C_{225;\delta} = C^3 a_5 \delta_{2\delta}, & C_{444;\delta} = 0, & C_{334;\delta} = 0, \\ C_{335;\delta} = 0, & C_{455;\delta} = 0, & C_{555;\delta} = 0, & C_{445;\delta} = 0. \end{array} \tag{5.1}$$

Putting (5.1) into (2.10), we get

$$\begin{array}{ll} (LC)_{;\delta} = C^3(\rho LC + 3a_2)\delta_{2\delta} + C^3 a_\delta, & LCu_\delta = C^3 a_3 \delta_{2\delta}, \\ LCv_\delta = C^3 a_4 \delta_{2\delta}, & LCw_\delta = C^3 a_5 \delta_{2\delta}. \end{array}$$

Since T_{hijk} is an indicatory tensor, from (C) it follows that $a_1 = a_i y^i = 0$. Thus, we have:

Theorem 5.1. If the *T*−tensor of a five-dimensional Finsler space is of the form (*C*), the scalar components a_α of the La_i are given by

$$\begin{array}{ll} a_1 = 0, \quad a_2 = \dfrac{L}{4}(C^{-3}C_{;2} - \rho C), & a_3 = LC^{-1}u_2 = C^{-3}(LC)_{;3}, \\ a_4 = LC^{-2}v_2 = C^{-3}(LC)_{;4}, & a_5 = LC^{-2}w_2 = C^{-3}(LC)_{;5}. \end{array}$$

Theorem 5.2. In a five-dimensional Finsler space with *T*−tensor of the form (*C*), the scalar components of *v*−connection vectors u_i, v_i and w_i are given by

$$LCu_\delta = C^3 a_3 \delta_{2\delta}, \quad LCv_\delta = C^3 a_4 \delta_{2\delta}, \quad LCw_\delta = C^3 a_5 \delta_{2\delta}.$$

Corollary 5.1. In a five-dimensional Finsler space with *T*−tensor of the form (*C*), the *v*−connection vectors u_i, v_i and w_i vanish if the scalar components a_3, a_4 and a_5 of La_i vanish.

VI. T–2 LIKE FINSLER SPACE

A non-Riemannian Finsler space $F^n (n > 2)$ is called *T*−2 like Finsler space if the *T*−tensor T_{hijk} is written in the form

$$T_{hijk} = \rho C_h C_i C_j C_k. \tag{6.1}$$

Theorem 6.1. In a *T*−2 like five-dimensional Finsler space, the *v*−connection vectors u_i, v_i and w_i vanish.

Theorem 6.2. In a *T*−2 like five-dimensional Finsler space, ρ is given by

$$\rho = C^{-4} C_{;2}.$$

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