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Some properties of the signless Lah numbers

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ABSTRACT: *In this paper, we obtain the ordinary generating function of the signless Lah numbers L(n, k) having only real zeros, and further derive the asymptotic normality of L(n, k). At the same time, we also prove the total positivity of the signless Lah triangle and the strict total positivity of the signless Lah square.*

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I. INTRODUCTION

Let $a(n,k)$ be a double-indexed sequence of nonnegative numbers and let

$$p(n,k)=\frac{a(n,k)}{\sum_{j=0}^{n}a(n,j)} \qquad (1)$$

denote the normalized probabilities. Following Bender [2], we say that the sequence $a(n,k)$ is asymptotically normal by a central limit theorem, if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\sum_{k\le\mu_n+x\sigma_n}p(n,k)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt\right|=0, \qquad (2)$$

where μ_n and σ_n^2 are the mean and variance of (1), respectively. We say $a(n,k)$ is *asymptotically normal by a local limit theorem* on $\mathbb{R}$ if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\sigma_n p(n,\lfloor\mu_n+x\sigma_n\rfloor)-\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right|=0. \qquad (3)$$

In this case,

$$a(n,k)\sim\frac{e^{-x^2/2}\sum_{j=0}^{n}a(n,j)}{\sigma_n\sqrt{2\pi}},\ \text{as } n\to\infty,$$

where $k=\mu_n+x\sigma_n$ and $x=O(1)$. Clearly, the validity of (2) implies that of (3).

Many combinatorial sequences have been proved to be asymptotic normality. In 1967, Harper [9] used the central limit theorem to prove that the Stirling numbers of the second kind are asymptotic normality. Recently, Wang, Zhang and Zhu [21] deduced that the Laplacian coefficients are approximately normal distributed if the size of a graph is large compared to the maximum degree. More results related to asymptotic normality in combinatorics can be referred to [1, 7, 10, 13, 16, 19].

In combinatorics, the signless Lah numbers, were introduced by Ivan Lah in 1952 and usually denoted by $L(n,k)$, count the number of n elements can be partitioned into k nonempty linearly ordered subsets and have an explicit formula $L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1}$ for all $n,k\in N$. Clearly, $L(n,0)=\delta(n,0)$ and $L(0,k)=\delta(0,k)$ for all $n,k\in N$.And the signless Lah numbers mainly originated [12] as connection constants:

$$x^{\bar{n}}=\sum_{k=0}^{n}L(n,k)x^{\underline{k}}. \qquad (4)$$

The signless Lah numbers can be represented in terms of Stirling numbers:

$$L(n,k)=\sum_{j=k}^{n}s(n,j)S(j,k),$$

where $s(n,j)$ and $S(j,k)$ are Stirling numbers of the first and second kinds, respectively. And we also find the generating fuction of the signless Lah numbers for all $k\in N$ is

$$\sum_{n\geq 0}L(n,k)\frac{x^n}{n!}=\frac{1}{k!}\left(\frac{x}{1-x}\right)^k. \qquad (5)$$

And the signless Lah numbers have many other interesting applications in analysis and combinatorics [4, 5, 8, 17].

Define the Lah matrix $LM_m=[L(n,k)]_{n,k\geq 0}$ as the matrix of dimension $m\times m$, whose elemnet in the n- th row and k- th column is $L(n,k)$. Note that LM_m is a low- triangular matrix, thus we also call $LM_m=[L(n,k)]_{n,k\geq 0}$ the Lah triangle, and we denote the Lah square by $LM_m^{\ulcorner}=[L(n+k,k)]_{n,k\geq 0}$. Let us recall some notation and definitions, we say that a matrix is totally positive (TP) if its minors of all orders are nonnegative. We say that a matrix is strictly totally positive (STP) if all its minors are positive. For example, the Pascal triangle is a TP matrix[11] . Total positivity plays an important role in many areas [6, 11, 18].

The organization of the paper is as follows: In Section 2, we mainly prove the asymptotic normality of the signless Lah numbers. In Section 3, we derive the total positivity of the signless Lah triangle LM_m and the strict total positivity of the signless Lah square $LM_m^{\ulcorner}$.

II. ASYMPTOTIC NORMALITY OF THE SIGNLESS LAH NUMBERS

Firstly, we give a standard approach to demonstrate asymptotic normality, which was used by Harper to show normality of the Stirling number of the second number.Many excellent results of asymptotic normality have been extensively studied, see [2, 3, 5] for instance.

Theorem2.1. ([9])Suppose that $A_n(x)=\sum_{k=0}^{n}a(n,k)x^k$ has only real roots, and $A_n(x)=\prod_{i=1}^{n}(x+r_i)$, where all the $a(n,k)$ and r_i are nonnegative. Let

$$\mu_n=\sum_{i=1}^{n}\frac{1}{1+r_i}$$

and

$$\sigma_n^2=\sum_{i=1}^{n}\frac{r_i}{(1+r_i)^2}.$$

Then if $\sigma_n\to+\infty$, the numbers $a(n,k)$ are asymptotically normal with the mean μ_n and variance σ_n^2.

Remark 2.2.([3]) When a polynomial $A_n(x)=\sum_{n\geq 0}a(n,k)x^k$ has real and nonpositive roots, we have a probabilistic interpretation of its coefficients. A bit of algebra shows that the mean and variance of random variables are given be the followling "rootfree" expressions

$$\mu_n = \frac{A_n'(1)}{A_n(1)}, \quad \sigma_n^2 = \frac{A_n'(1)}{A_n(1)} + \frac{A_n''(1)}{A_n(1)} - \left(\frac{A_n'(1)}{A_n(1)}\right)^2. \tag{6}$$

We can use this expression of mean and variance to deduce the results of the above theorem.

Lemma 2.3.([20]) The sequence L(n, k) satisfy the three-term recurrence

$$L(n+1,k) = L(n,k-1) + (n+k)L(n,k). \tag{7}$$

Then, let

$$L_n(x) = \sum_{k=0}^{n} L(n,k)x^k.$$

We can use Lemma 2.3 to obtain the following theorem.

Theorem 2.4. The polynomial sequence $L_n(x)$ satisfy the three- term recurrence:

$$L_n(x) = (x+n-1)L_{n-1}(x) + L_{n-1}'(x), \quad L_0(x) = 1. \tag{8}$$

Proof. We multiply x^k on both side of the recurrence of the signless Lah numbers and sum over the values of k, this yields

$$\begin{aligned} L_n(x) &= \sum_{k=0}^{n} L(n-1,k-1)x^k + \sum_{k=0}^{n} L(n-1,k)x^k \\ &= (x+n-1)\sum_{k=0}^{n-1} L(n-1,k)x^k + x\sum_{k=0}^{n-1} kL(n-1,k)x^{k-1}. \end{aligned}$$

Then we compare the two sides of the equal sign, we obtain the desired one.

Theorem 2.5 ([14]). Let f and g be a real polynomials whose leading coefficients have the same sign. Suppose that f,g ∈ RZ and g ≼ f, if ad ≤ bc, then (ax+b)f(x)+(cx+ d)g(x) ∈ RZ.

Corollary 2.6 ([14]). Suppose that f,g ∈ PF and g interlaces f, if ad ≥ bc, then (ax + b)f(x) + x(cx + d)g(x) ∈ RZ.

The above theorem and corollary provide the inductive basis for the reality of zeros of polynomials sequence $P_n(x)$ satisfying certain recurrence relations $P_n(x) = a_n(x)P_{n-1}(x) + b_n(x)P_{n-1}'(x)$. Thus for the recurrence relation (2.3) satisfying the Theorem 2.5, we can immediately get the following theorem.

Theorem 2.7. The polynomial $L_n(x)$ has only real zeros for each $n \geq 1$.

Finally, we give the main result of this section.

Theorem 2.8. The sequence L(n, k) are asymptotically normal.

Proof. We let $\sum_{k=0}^{n} L(n,k) = L_{n,s}$. Applying (6) to $L_n(x) = \sum_{k=0}^{n} L(n,k)x^k$, we obtain

$$\mu_n = \frac{\sum_{k=0}^{n} kL(n,k)}{L_{n,s}}, \tag{9}$$

$$\sigma_n^2 = \frac{\sum_{k=0}^{n} k^2L(n,k)}{L_{n,s}} - (\mu_n)^2. \tag{10}$$

According to the recurrence relation of $L(n,k)$, after calculation, we can obtain

$$\sum_{k=0}^{n} kL(n,k) = L_{n+1,s} - (n+1)L_{n,s}, \tag{11}$$

$$\sum_{k=0}^{n} k^2L(n,k) = L_{n+2,s} - (2n+3)L_{n+1,s} + n(n+2)L_{n,s}. \tag{12}$$

Thus, the (9) and (10) can be expressed

$$\mu_n = \frac{L_{n+1,s}}{L_{n,s}} - (n+1), \tag{13}$$

$$\sigma_n^2 = \frac{L_{n+2,s}}{L_{n,s}} - \frac{L_{n+1,s}}{L_{n,s}} - \left(\frac{L_{n+1,s}}{L_{n,s}}\right)^2 - 1. \tag{14}$$

Since $\sum_{n\geq 0} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}(\frac{x}{1-x})^k,\ k \in N$ and $\sum_{k=0}^{n} L(n,k) = L_{n,s}$, Thus we can get the generating function of the $L_{n,s}$ by

$$\sum_{n=0}^{\infty} L_{n,s}\frac{x^n}{n!} = \exp\left(\frac{x}{1-x}\right). \tag{15}$$

Then, we calculate the asymptotic formula of the $L_{n,s}$ by Cauchy's formula applied to (15) gives

$$L_{n,s} = \frac{n!}{2\pi i}\int_{\tau} \frac{\exp(\frac{x}{1-x})}{x^{n+1}}dx. \tag{16}$$

We let $x = Re^{i\theta}$, where R will be determined later. Then (16) becomes

$$\begin{aligned} L_{n,s} &= \frac{n!}{2\pi R^n}\int_{-\pi}^{\pi} \exp\left(\frac{R\cos\theta + Ri\sin\theta}{1 - R(\cos\theta + i\sin\theta)} - in\theta\right)d\theta \\ &= \frac{n!}{2\pi R^n}\int_{-\pi}^{\pi} \exp\left(\frac{R\cos\theta + Ri\sin\theta}{1 - R(\cos\theta + i\sin\theta)} - in\theta\right)d\theta. \end{aligned}$$

We decompose this last integral into three parts

$$(\int_{-\pi}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\pi}) \exp\left(F(\theta)\right)d\theta,$$

with $F(\theta) = \frac{R(\cos\theta + i\sin\theta)}{1-R(\cos\theta+i\sin\theta)} - in\theta = \frac{R\cos\theta + Ri\sin\theta}{1-R\cos\theta - Ri\sin\theta} - in\theta$ and $\varepsilon = n^{-\frac{1}{4}}$, we prove that the integrals $\int_{-\pi}^{-\varepsilon}$ and $\int_{\varepsilon}^{\pi}$ are negligible, and then the greatest contribution come from the origin. First of all by differentiating θ, we can get

$$\begin{aligned} F'(\theta) &= -(1 - R\cos\theta - Ri\sin\theta)^{-2}(R\sin\theta - Ri\cos\theta) - in, \\ F''(\theta) &= -(-2(1 - R\cos\theta - Ri\sin\theta)^{-3}(R\sin\theta - Ri\cos\theta)^2 \\ &\quad +(1 - R\cos\theta - Ri\sin\theta)^{-2}(R\cos\theta + Ri\sin\theta)). \end{aligned}$$

Then

$$\begin{aligned} F(0) &= \frac{R}{1-R}, \\ F''(0) &= -\frac{R}{(1-R)^2} - \frac{2R^2}{(1-R)^3}. \end{aligned}$$

We choose R as the only solution of $F'(0) = 0$ that is greater than zero and less than one: $\frac{R}{(1-R)^2} = n$, thus $F''(0) = -[n + \frac{2R^2}{(1-R)^3}]$. Then expanding the integrand in a Taylor series about $\theta = 0$, we obtain

$$\begin{aligned} \left|\int_{\varepsilon}^{\pi} \exp\{F(\theta)\}d\theta\right| &\leq \int_{\varepsilon}^{\pi} \exp\left(\frac{R}{1-R} - \frac{\theta^2}{2}\left(n + \frac{2R^2}{(1-R)^3}\right) + o(\theta^2)\right)d\theta \\ &\leq \exp(\frac{R}{1-R})\int_{\varepsilon}^{\pi} \exp\left(-\frac{\theta^2}{2}\left(n + \frac{2R^2}{(1-R)^3}\right) + o(\theta^2)\right)d\theta. \end{aligned}$$

The integral in the last expression is

$$\int_{\varepsilon}^{\pi} \exp\left(-\frac{\theta^2}{2}\left(n + \frac{2R^2}{(1-R)^3}\right) + o(\theta^2)\right)d\theta \to 0, \quad n \to \infty.$$

The same calculation is valid for $\int_{-\pi}^{-\varepsilon}$. Finally, we obtain

$$L_{n,s} \sim \frac{n!}{2\pi R^n}\exp(\frac{R}{1-R})\int_{-\varepsilon}^{\varepsilon} \exp\left(-\frac{\theta^2}{2}\left(n + \frac{2R^2}{(1-R)^3}\right) + o(\theta^2)\right)d\theta. \tag{17}$$

Putting $\psi = \sqrt{n + \frac{2R^2}{(1-R)^3}}\theta$ in (17) and observing that for n large enough, we can integrate on the real axis. We obtain

$$\begin{aligned} L_{n,s} &\sim \frac{n!}{2\pi R^n \sqrt{n + \frac{2R^2}{(1-R)^3}}} \exp\left(\frac{R}{1-R}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{\psi^2}{2}\right) d\psi \\ &\sim \frac{n!}{R^n \sqrt{2\pi[n + \frac{2R^2}{(1-R)^3}]}} \exp\left(\frac{R}{1-R}\right). \end{aligned} \tag{18}$$

Using this last estimate of estimate of asymptotic formula of the $L_{n,s}$ and plugging the (14), we can obtain

$$\begin{aligned} \sigma_n^2 &= \frac{L_{n+2,s}}{L_{n,s}} - \frac{L_{n+1,s}}{L_{n,s}} - (\frac{L_{n+1,s}}{L_{n,s}})^2 - 1 \\ &= \frac{(n+2)(n+1)}{R^2}\sqrt{1 - \frac{2}{n + \frac{2R^2}{(1-R)^3} + 2}} - \frac{(n+1)(n+1)}{R^2}(1 - \frac{1}{n + \frac{2R^2}{(1-R)^3} + 1}) \\ &\quad - \frac{n+1}{R}\sqrt{1 - \frac{1}{n + \frac{2R^2}{(1-R)^3} + 1}} - 1 \\ &= \frac{(n+1)(1-R)}{R^2} - 1, \qquad n \to \infty. \end{aligned}$$

Thus, $\sigma_n^2 \to \infty$ as $n \to \infty$. By Theorem 2.1, this finally proves the main theorem.

III. THE SIGNLESS LAH TRIANGLE AND THE SIGNLESS LAH SQUARE

We now prove the total positivity of the signless Lah triangle and the strict total positivity of the signless Lah square due to the method [22].Let $A = [a_{n,k}]$ and $B = [b_{n,k}]$, we define $a_{n,k} \sim b_{n,k}$ and $A \sim B$ when there exist positive numbers x_n and y_k such that $b_{n,k} = x_n y_k a_{n,k}$. Thus we can easily get the following proposition by definition.

Proposition 3.1. If A ~ B, Then the matrix A is TP (resp. STP) if and only if the matarix B is TP (resp. STP)

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers, then its Toeplitz matrix is

$$[a_{n-k}]_{n,k\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

A sequence $(a_n)_{n\geq 0}$ is $Pólya frequency$ (PF) if its Toeplitz matrix is TP.The polynomial $L_n(x)$ has only real zeros for each $n \geq 1$.

Theorem 3.2 ([11]).A nonnegative sequences $(a_n)_{n\geq 0}$ is PF if and only if its generating function has the form

$$\sum_{n\geq 0} a_n x^n = \frac{\prod_j (1 + \alpha_j x)}{\prod_j (1 - \beta_j x)} e^{\gamma x},$$

where $\alpha_j, \beta_j, \gamma \geq 0$ and $\sum_j (\alpha_j + \beta_j) < +\infty$.

We can get the sequence $(\frac{1}{n!})_{n\geq 0}$ is PF by Theorem 3.2. thus its corresponding Toeplitz matrix $[a_{n-k}] = [\frac{1}{(n-k)!}]$ is TP. Then we can easily get the following Theorem.

Theorem 3.3. the signless Lah triangle LM_m is TP.

Proof. We have

$$LM_m = \frac{n!}{k!}\binom{n-1}{k-1} = \frac{n!(n-1)!}{k!(k-1)!(n-k)!} \sim \frac{1}{(n-k)!}.$$

We can get the signless Lah triangle is TP due to the matrix $[\frac{1}{(n-k)!}]$ is TP, thus we get the desired result.

Let $(b_n)_{n\geq 0}$ is a sequence with real numbers, and its Hankel matrix is

$$[b_{n+k}]_{n,k\geq 0} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ b_1 & b_2 & b_3 & b_4 & \cdots \\ b_2 & b_3 & b_4 & b_5 & \cdots \\ b_3 & b_4 & b_5 & b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

we say that the sequence $(b_n)_{n\geq 0}$ is a $Stieltjesmoment$ (SM) sequence if it can be expressed as

$$b_n = \int_0^{+\infty} x^n d\lambda(x),$$

where λ is a nonnegative measure on $[0, +\infty]$.

Lemma 3.4. ([18]) A sequence $(a_n)_{n\geq 0}$ is SM if and only if the Hankle matrix $[a_{i+j}]$ is STP; or both $[a_{i+j}]_{0\leq i,j\leq n}$ and $[a_{i+j+1}]_{0\leq i,j\leq n}$ are positive definite.

We can get the sequence $(n!)_{n\geq 0}$ is a SM sequence since

$$n! = \int_{0+\infty} x^n e^{-x} dx = \int_{0+\infty} x^n d(1-e^{-x}),$$

thus its corresponding Hankle matrix $[(n+k)!]$ is STP by Lemma 3.4. Then we have the following results.

Theorem 3.5. The signless Lah square $LM_m^{\ulcorner}$ is STP.

Proof. We have

$$L(n+k,k) = \frac{(n+k-1)!(n+k)!}{n!(k-1)!k!} \sim (n+k-1)!(n+k)!.$$

So, to show the Lah square $LM_m^{\ulcorner}$ is STP, it suffices to show that the sequence $(n!(n+1)!)_{n\geq 0}$ is SM. If a matrix is STP, we can get its submatrix is still STP. Hence if the sequence $(b_n)_{n\geq 0}$ is SM, its shifted sequence $(b_{n+1})_{n\geq 0}$ is also SM by Lemma 3.4. Thus the sequence $(n!)_{n\geq 0}$ and the sequence $((n+1)!)_{n\geq 0}$ are all SM. Then we state that the Hadamard product of two positive definite matrices is still positive definite (see [18] for details). We have $(a_n b_n)_{n\geq 0}$ is SM if $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are all SM by Lemma3.4. Thus we obtain $(n!(n+1)!)_{n\geq 0}$ is SM, the signless Lah square is STP.

IV. REMARKS

The s- associated Lah numbers, which express the number of partitions of Z_n into k order list contains at least s, denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^s$. The s- associated Lah numbers satisfy the following recurrence relation, for $n \geq sk$,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^s = \binom{n-1}{k-1} s! \left\{ \begin{matrix} n-s \\ k-1 \end{matrix} \right\}^s + (n+k-1) \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}^s.$$

We can easily get the asymptotic normality of the 2- associated Lah numbers with the same method as Theorem 2.8, and we proposed the following conjecture.

Conjecture 4.1. The s-associated Lah numbers are asymptotically normal.

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REFERENCES

[1]. P. Baldi, Y. Rinott, Asymptotic normality of some graph-related statistics, J. Appl. Prob. 26 (1989) 171–175.

[2]. E.A. Bender, Central and local limit theorem applied to asymptotic enumeration, J. Combin. Theory Ser. A 15 (1973) 91–111.

[3]. E.R. Canfield, Asymptotic normality in enumeration, Handbook of enumerative combinatories, 255–280, Discrete math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015.

[4]. C.A. Charalambides, Enumerative Combinatoric, CRC Press Series on Discrete Mathematics and its Applications. Chapman and Hall/CRC, Boca Raton, FL, 2002.

[5]. L. Comtet, Adanced Combinatorics, Reidel, Dordrecht, 1974.

[6]. S.M. Fallat, C.R. Johanson, Totally Nonnegative matrices, Princeton University Press, Princeton, 2011.

[7]. D. Galvin, Asymptotic normality of some graph sequences, Graphs Combin. 22 (2016) 639–647.

[8]. R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics-A Foundation for Computer Science,2nd ed, Addison-Wesley Publishing Company, Reading, MA, 1994.

[9]. L.H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Stat. 38 (1967) 410–414.

[10]. J. Kahn, A normal law for matchings, Combin. 20 (2000) 339–391.

[11]. S. Karlin, Total Positivity, Volume 1, Stanford University Press, 1968.

[12]. I. Lah, Eine neue art von zahlen, ihre eigenschaften und anwendung in der math- ematischen statistik, Mitteilungsbl, Math. Statist. 7 (1955) 203–212.

[13]. J.L. Lebowitz, B. Pittel, D.Ruelle, E.R. Speer, Central limit theorems, Lee-Yang zeros, and graph-counting polynomials, J. Combin. Theory Ser. A 141 (2016) 147– 183.

[14]. L.L. Liu, Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Apple. Math. 38 (2007) 542–560.

[15]. T.S. Motzkin, Sorting numbers for cylinders other classification numbers, Proc. Symp. Pure Math, Vol. 19(American Mathematical Society, Providence, RI, 1971) 167–176.

[16]. K. Nowicki, Asymptotic normality of graph statistics, J. Stat. Plan. Infer. 21 (1989) 209–222.

[17]. M. Petkovˇsek, T. Pisanski, Combinatorial interpretation of unsigned Stirling and Lah numbers, Preprint 40 (2002), Univ. of Ljubljana.(Available on the Internet).

[18]. A. Pinkus, Totally positive matrices, Cambridge University Press, Cambridge, 2010.

[19]. A. Rucinski, The behaviour of $\frac{\binom{n}{k,\ldots k,n-ik}}{i!}$ is asymptotically normal, Discrete Math.49 (1984) 287–290.

[20]. C. Wagner, Generalized Stirling and Lah numbers, Discrete Math. 160 (1996) 199–218.

[21]. Y. Wang, H.-X. Zhang, B.-X. Zhu, Asymptotic normality of Laplacian coefficients of graphs, J. Math. Anal. Appl. 455 (2017) 2030–2037.

[22]. Y. Wang, A.L.B. Yang, Total positivity of Narayana matrices, Discrete Math. 341 (2018) 1264–1269.