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Lie Symmetry Method to Solve Linear Ordinary Differential Equation of First Order

Nidhi Sharma[1] and Dr Gaurav Kumar[2]
[1]*Research Scholar, SBAS, Shobhit Institute of Engineering & Technology (Deemed to be University), Meerut, U.P., India*
[2]*Associate Professor, Department of Mathematics, N.A.S. College, Meerut, U.P., India*

Abstract: *Basis of Lie symmetry method is finding of similarity of ordinary differential equations. These symmetries of the equations are then used to find general solutions of the equations. The present paper discusses Lie Similarity Method for first order ODE (ordinary differential equation) admitting Lie group symmetry. First order linear ODE has been solved using Lie Symmetry Method. In this work, examples of first order linear ODEs are considered to explain the method.*
Keywords: *Lie Symmetry Method, Similarity Solutions, Linear ODE, Lie Invariance*

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I. Introduction

In sixties of 19th Century, Sophus Lie, a Norwegian Mathematician, developed techniques of point symmetries to solve differential equations. He discovered the relationship between group theory and conventional methods of finding solutions/solution curves for ordinary differential equations. Symmetries of differential equations are defined as the transformations of the dependent variable as well as the independent variable along with the derivatives of dependent variable. The transformations map solutions of differential equation to that of transformed equation. Therefore knowledge of transformations of a differential equation may help in finding its solutions or to transform it to a form which is easily integrable. In case of ordinary differential equations, these transformations help to reduce order of differential equation or to find integrating factor. A transformation of derivative of a dependent variable which is induced by the transformation of the dependent and independent variable is called point transformation and the corresponding symmetries of the differential equation are called point symmetries. Lie Group symmetry is defined as a continuous transformation which takes each solution to another solution.

II. Literature Review

Olver and Rosenau (1987) studied group invariant solutions of differential equations. Algebraic equation is solved by **Bluman (1990).** Bluman then found solutions of an ordinary differential equation using infinitesimals of an admitted Lie group of transformations which were invariant. Integrability of nonlinear ordinary differential equations were studied by **Paliathanasis and Leach (2016).** They discussed the symmetry analysis and singularity analysis methods. **Sahoo et al. (2017)** found exact solutions of the modified KdV–Zakharov–Kuznetsov equation by using Lie group analysis. The theory of systems of ordinary differential equations of first order being extended by **Carinena et al. (2018). Dorodnitsyn et al. (2018)** analysed first order delay ordinary differential equations and presented its group classification. **Hu and Du (2019)** obtained necessary conditions of the existence of the first integrals of second order ordinary differential equations and then used these integrals to obtain first integrals of the equation using Lie symmetry method. **Lobo and Valaulikar (2019)** found symmetries of first order non-homogeneous neutral differential equations with variable coefficients using Taylor's theorem. They split this equation to find a system of partial differential equations which were over-damped. They then solved this system to obtain corresponding infinitesimals. These infinitesimals are then used to find equivalent symmetries. **Mekheimer et al. (2020)** analysed equations of two dimensional incompressible steady fluid flow with the heat transfer using Lie symmetries.

III. Lie Symmetry Method

In this paper, Lie Symmetries for first order linear ordinary differential equations are formulated. These symmetries are applied to formulate similarity solutions of order ordinary differential equations. Outlines of Lie symmetry method is presented here for ODE of first order. First order ODE is given by –

$$\frac{dx}{dt} = F(t, x), \tag{1}$$

where *x* dependent and t is independent variable.
Consider Lie group of one parameter

$$\bar{t} = f(t, x; \varepsilon), \ \bar{x} = g(t, x; \varepsilon). \tag{2}$$

If (1) is assumed invariant under transformation (2), then –

$$\frac{d\bar{x}}{d\bar{t}} = F(\bar{t}, \bar{x}), \tag{3}$$

which gives

$$\frac{g_t + g_x F(t, x)}{f_t + f_x F(t, x)} = F\left(f(t, x, \varepsilon), g(t, x, \varepsilon)\right). \tag{4}$$

To solve (4), extended form of Lie group called infinitesimal transformations were considered by Lie. The infinitesimals which transform (1) are given by the solutions of Lie's invariance condition –

$$X_t + (X_x - T_t)F - T_x F^2 = TF_t + XF_x \tag{5}$$

From the solutions of (5) we find the infinitesimals *X* and *T*.
Using these infinitesimals *X* and *T*, we transform (1) to canonical coordinates *(r, s)* by -

$$r_t T + r_x X = 0 \tag{6}$$

$$s_t T + s_x X = 1 \tag{7}$$

Lagrange's method of characteristics can be used to solve the equations (6) & (7) so as to find *r* and *s*. We have then -

$$\frac{ds}{dr} = \frac{s_t + s_x \frac{dx}{dt}}{r_t + r_x \frac{dx}{dt}} \tag{8}$$

Equation (8) is solved by usual method of solving first order ODE to find the general solution of equation (1).

IV. Examples

In this section, Lie symmetry method for liner ODE of Ist order is explained with the help of examples.
Example 4.1 Consider first order ODE -

$$\frac{dx}{dt} = 2x + t \tag{9}$$

Here *x* is dependent and *t* is independent variable. Equation (9) is a linear ODE in dependent variable *x*.

Comparing equation(9) with $\frac{dx}{dt} = F(t, x)$, we get

$$F(t, x) = 2x + t \tag{10}$$

Lie invariance condition is given by

$$X_t + (X_x - T_t)F - T_x F^2 = TF_t + XF_x \tag{11}$$

where $X \ and \ T$ are infinitesimals to be found.
From equation (10), we get

$$F_x = 2, F_t = 1 \tag{12}$$

Putting these values in Lie's invariance condition (11), we get -

$$X_t + (X_x - T_t)(2x + t) - T_x(2x + t)^2 = T.1 + X.2$$

$$X_t + (X_x - T_t)(2x + t) - T_x(4x^2 + t^2 + 4xt) = T + 2X \tag{13}$$

To find X and T, we shall equate various powers of x on both sides of (13).
Coefficient of x^2 :

$$-4T_x = 0$$

$T_x = 0$ (14)

$T = A(t)$ (15)

where $A(t)$ is same function of t or constant.

Coefficient of x :

$2X_x - 2T_t - 4tT_x = 0$ (16)

Using equation (14) in (16), we get -

$2X_x - 2T_t = 0$

$X_x = T_t$ (17)

Coefficient of x^0 :

$X_t + tX_x - tT_t = T + 2X$

Using equation (17), we get

$X_t = T + 2X$ (18)

If we take $A(t)$ to be zero then from equation (15), we get

$T = 0$ (19)

From equation (18), we get

$X_t = 2X$

$$\frac{dX}{dt} = 2X$$

$$\frac{dX}{X} = 2dt$$

$\log X = 2t + \log B$

$X = Be^{2t}$

where B is constant of integration. We take B =1, we get

$X = e^{2t}$

Therefore infinitesimals X and T are given by

$X = e^{2t}, T = 0$ (20)

Now we find canonical co-ordinates r and s .

We use following equations to find r and s

$r_t T + r_x X = 0$ (21)

$s_t T + s_x X = 1$ (22)

From equation (21), we get

$r_t.0 + r_x e^{2t} = 0$

Since $e^{2t} \neq 0$ therefore $r_x = 0 \Rightarrow r = t$ (23)

From equation (22), we get

$s_t.0 + s_x e^{2t} = 1$

$$s_x = \frac{1}{e^{2t}}$$

$$s = \frac{x}{e^{2t}} \quad (24)$$

$$\therefore \quad (r(t,x), s(t,x)) = (t, \frac{x}{e^{2t}})$$

From equation (23) and (24), we have

$$r_t = 1, \quad r_x = 0 \tag{25}$$

$$s_t = -2xe^{-2t}, \quad s_x = e^{-2t} \tag{26}$$

Now substituting $r \; and \; s$ into

$$\frac{ds}{dr} = \frac{s_t + s_x \dfrac{dx}{dt}}{r_t + r_x \dfrac{dx}{dt}}$$

We get

$$\frac{ds}{dr} = \frac{-2xe^{-2t} + e^{-2t}(2x+t)}{1+0}$$

$$\frac{ds}{dr} = -2xe^{-2t} + e^{-2t}(2x+t)$$

$$= -2xe^{-2t} + 2xe^{-2t} + te^{-2t}$$

$$\frac{ds}{dr} = te^{-2t} \tag{27}$$

Since from equation (23), $t = r$

Therefore from (27), we get

$$\frac{ds}{dr} = re^{-2r}$$

$$ds = re^{-2r}dr$$

Integrating

$$s = \int re^{-2r}dr + c$$

where c is the constant of integration.

$$s = \frac{re^{-2r}}{-2} - 1.\frac{e^{-2r}}{(-2)(-2)} + c$$

$$s = \frac{-1}{2}re^{-2r} - \frac{1}{4}e^{-2r} + c$$

$$s = \frac{-1}{2}(r + \frac{1}{2})e^{-2r} + c \tag{28}$$

Using equation (23) and (24), we get

$$r = t, \; s = \frac{x}{e^{2t}} = xe^{-2t}$$

Substituting in equation (28), we get

$$xe^{-2t} = \frac{-1}{2}(t + \frac{1}{2})e^{-2t} + c$$

$$x = \frac{-1}{2}(t + \frac{1}{2}) + ce^{2t} \tag{29}$$

which is the general solution of given equation (9).

Example 4.2 Consider first order ODE **-**

$$\frac{dx}{dt} = x - t \tag{30}$$

Here x is dependent and t is independent variable. Equation (30) is a linear ODE in dependent variable x.

Comparing equation(9) with $\frac{dx}{dt} = F(t,x)$, we get

$$F(t,x) = x - t \tag{31}$$

Lie invariance condition is given by

$$X_t + (X_x - T_t)F - T_x F^2 = TF_t + XF_x \tag{32}$$

where $X \ and \ T$ are infinitesimals to be found.

From equation (31), we get

$$F_x = 1, F_t = -1 \tag{33}$$

Putting these values in Lie's invariance condition (32), we get -

$$X_t + (X_x - T_t)(x-t) - T_x(x-t)^2 = T.1 - X$$

$$X_t + (X_x - T_t)(x-t) - T_x(x^2 + t^2 - 2xt) = T - X \tag{34}$$

To find X and T , we shall equate various powers of x on both sides of (34).

Coefficient of x^2 :

$$-T_x = 0$$

$$T_x = 0 \tag{35}$$

$$T = A(t) \tag{36}$$

where $A(t)$ is same function of t or constant.

Coefficient of x :

$$X_x - T_t + 2tT_x = 0 \tag{37}$$

Using equation (35) in (37), we get -

$$X_x - T_t = 0$$

$$X_x = T_t \tag{38}$$

Coefficient of x^0 :

$$X_t - tX_x + tT_t = T - X$$

Using equation (38), we get

$$X_t = T - X \tag{39}$$

If we take $A(t)$ to be zero then from equation (36), we get

$$T = 0 \tag{40}$$

From equation (39), we get

$$X_t = -X$$

$$\frac{dX}{dt} = -X$$

$$\frac{dX}{X} = -dt$$

$$\log X = -t + \log B$$

$$X = Be^{-t}$$

where B is constant of integration. We take B =1, we get

$$X = e^{-t}$$

Therefore infinitesimals X and T are given by

$$X = e^{-t}, T = 0 \tag{41}$$

Now we find canonical co-ordinates $r \ and \ s$.

We use following equations to find $r \ and \ s$

$$r_t T + r_x X = 0 \tag{42}$$

$$s_t T + s_x X = 1 \tag{43}$$

From equation (21), we get

$$r_t .0 + r_x e^{-t} = 0$$

Since $e^{-t} \neq 0$ therefore $r_x = 0 \quad \Rightarrow \quad r = t$ (44)

From equation (43), we get

$$s_t .0 + s_x e^{-t} = 1$$

$$s_x = \frac{1}{e^{-t}}$$

$$\Rightarrow \quad s = \frac{x}{e^{-t}} \tag{45}$$

$$\therefore \qquad (r(t,x), s(t,x)) = (t, \frac{x}{e^{-t}})$$

From equation (23) and (24), we have

$$r_t = 1, \quad r_x = 0 \tag{46}$$

$$s_t = xe^t, \ \ s_x = e^t \tag{47}$$

Now substituting $r \ and \ s$ into

$$\frac{ds}{dr} = \frac{s_t + s_x \frac{dx}{dt}}{r_t + r_x \frac{dx}{dt}}$$

We get

$$\frac{ds}{dr} = \frac{xe^t + e^t(x-t)}{1+0}$$

$$\frac{ds}{dr} = xe^t + e^t(x-t)$$

$$= xe^t + xe^t - te^t$$

$$\frac{ds}{dr} = 2xe^t - te^t \tag{48}$$

From (44) and (45) in (48), we get-

$$\frac{ds}{dr} = 2s - re^r$$

$$\frac{ds}{dr} - 2s = -re^r \tag{49}$$

General solution of (49) is given by –

$$se^{-2r} = \int re^r e^{-2r} dr + c$$

$$= \int re^{-r} dr + c$$

Which gives

$$s = c - (r+1)e^r$$

Putting value if *r* and *s* in terms of *x* and *y*, we get –

$$x = ce^{-t} - (t+1) \tag{50}$$

where *c* is the constant of integration.

(50) gives general solution of (30).

V. Conclusion

Lie symmetry method is one of the finest methods to solve differential equation. It has a systematic algorithm to solve first order ODE. The method can be applied to higher order ODE so as to reduce their order and then find general solution. This method obtains exact solution linear ODE of first order.

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