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Lie Symmetry Solution of Third Order Linear Ordinary Differential Equations

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ABSTRACT
This research paper consists solution of third order linear ordinary differential equations with Lie symmetry theory. There are various methods for solving differential equation, Lie symmetry is one of them. Lie symmetry is powerful method for finding the solution of differential equations. It gives exact solution of given differential equation. This method reduces the higher order ODE into linear or quadratic which is shown in this paper.
Keywords: *Lie symmetry Method, Third order ODE, Infinitesimal Generators etc.*

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I. INTRODUCTION

Differential equations is a major branch of applied mathematics, which has a vital role in physical sciences and engineering etc. There are various methods for solving ordinary differential equation if the differential equation fit in any of the form like as variable separable technique, homogeneous form, linear form, exact form etc. then it is solvable easily by any well known method. But if differential equation doesn't fit in any of the form then it's not easy to find it's exact solution. In that scenario, we find the solution of differential equations with the help pf numerical technique such as Euler's method, Picard method, Euler modified method, Jacobi method etc. but all these methods don't provide the exact solution these methods provide approximate solutions of differential equations. To get rid of this problem, Norwegian mathematician Sophus Lie (1842-1899) developed a concept by which we can get exact solution of given differential equations. Sophus Lie started to find the continuous group of transformations which remains differential equations unchanged, now this is called 'symmetry analysis of differential equations'. Lie symmetry method is very useful for solving differential equations. Lie symmetry method reduces the higher order ordinary differential equations into linear or quadratic form which is solvable easily by any well known method. In this research paper, we solved third order linear ordinary differential equation with the help of Lie symmetry method.

II. LITERATURE REVIEW

Olver and Rosenau (1987) studied about the solutions of differential equations. They introduced weak symmetry's concept for the systems of differential equations. **George Bluman (1990)** proved that for an ordinary differential equation, invariant solutions can be obtained by solving an algebraic equation which is obtained from given ordinary differential equation and infinitesimals. **P.J. Olver (1993)** worked for applications of Lie groups, he studied about some application based on Lie symmetry analysis. In this work, he told about symmetries and how the symmetries work in differential equation. **G. Baumann (1997)** worked with symmetry analysis of differential equations he did lot of work in this field, in this paper, he told about the symmetry of differential equation that how a differential equation can be solved by using symmetry analysis. **L. Dresner (1999)** gave the theory of differential equations. In this review, Dresner introduced the concept of Lie theory in partial differential equations also. He did that how partial differential equation which is not solvable by any well-known method can be solved by Lie symmetry theory. Lie symmetry method also help in mathematical models in epidemiology. **M.C. Nucci and V. Torrisi (2001)** worked with the theory of Lie group analysis to mathematical model which describes HIV transmission, in the geometrical study of differential equations. Lie symmetry method is versatile and powerful tool for solving differential equations with initial and boundary value problems. **P.E. Hydon (2005)** dealt with the initial value problems with the help of symmetry analysis. **J. Starrett (2007)** used the lie symmetry method for solving first and second order ordinary differential equations. He obtained the linearized symmetry condition for first order ordinary differential equations. In this research paper, he solved some non-linear ordinary differential equations with the use of this method. about lie reduction method for full symmetry classification of one parameter group invariant solutions of ordinary differential equations. **Mahomed and Qadir (2008)** extended the geometric methods for linearizing systems of second order cubic non linear in the first derivatives ordinary differential equations to the third order by differentiating the second order equation. **F. Oliveri (2010)** dealt with some classical results in lie symmetry theory. In his research papers, he told us to about infinitesimal generators and recent contributions of lie symmetry theory. **Pue and Meleshko (2010)** gave complete Lie group classification of second order delay differential equation. They presented all classes of such equations admitting Lie algebra. **R. Mohanasubha and M. Senthilvelan (2015)** worked with the non-linear ordinary differential equations. They told that how to solve non-linear ordinary differential equation by using symmetry analysis. **Hu and Du (2019)** started first inegrals of a second order ordinary differential equations. They presented necessary conditions of existence of first inegrals for the equation and then obtained first integrals of the equation using Lie symmetry method. **W.Khalid Jaber and Sajid Mohammad (2020)** worked with the Lie symmetry theory. They solved first and second order linear ordinary differential equations with help of Lie symmetry. **N. Sharma and G. Kumar (2022)** solved bernoulli's differential equation of first order with the help of Lie symmetry method.

III. EXAMPLE

We shall consider a third order ODE

$$G\left(t, x, x', x'', x'''\right) = 0$$

Or

$$x''' = g\left(t, x, x', x''\right)$$

we shall take a special case of third order ODE such as

$$x''' = 0 \tag{1}$$

we shall use third extension of $S_{[3]}$, thus we have

$$S_{[3]} = S_{[2]} + \left(\gamma''' - 3x'''\delta' - 3x''\delta'' - x'\delta'''\right)\frac{\partial}{\partial x'''}$$

$$S_{[3]} = S_{[1]} + \left(-x'\delta'' - 2x''\delta' + \gamma''\right)\frac{\partial}{\partial x''} + \left(-x'\delta''' - 3x''\delta'' - 3x'''\delta' + \gamma'''\right)\frac{\partial}{\partial x'''}$$

$$S_{[3]} = \gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial t} + \left(-x'\delta' + \gamma'\right)\frac{\partial}{\partial x'} + \left(-x'\delta'' - 2x''\delta' + \gamma''\right)\frac{\partial}{\partial x''} + \left(-x'\delta''' - 3x''\delta'' - 3x'''\delta' + \gamma'''\right)\frac{\partial}{\partial x'''} \tag{2}$$

now we operate $S_{[3]}$ on equation (1), then we get

$$S_{[3]}\left[x'''\right] = 0$$

$$\left[\gamma x'''' + \delta.0 + \left(-x'\delta' + \gamma'\right).0 + \left(-x'\delta'' - 2x''\delta' + \gamma''\right).0 + \left(-x'\delta''' - 3x''\delta'' - 3x'''\delta' + \gamma'''\right).1\right] = 0$$

$$\left[\gamma x'''' + \left(-x'\delta''' - 3x''\delta'' - 3x'''\delta' + \gamma'''\right)\right] = 0 \tag{3}$$

Using equation (1) in equation (3), we have

$$\left[\gamma x'''' + \left(-x'\delta''' - 3x''\delta'' + \gamma'''\right)\right] = 0 \tag{4}$$

Differentiate equation (1) with respect to x, we get

$$x'''' = 0 \tag{5}$$

Using equation (5) in equation (4), we get

$$\left[\gamma.0 + \left(-x'\delta''' - 3x''\delta'' + \gamma'''\right)\right] = 0$$

$$\left[\left(-x'\delta''' - 3x''\delta'' + \gamma'''\right)\right] = 0 \tag{6}$$

The first, second and third derivatives of γ and δ are,

$$\gamma' = \frac{\partial\gamma}{\partial t} + x'\frac{\partial\gamma}{\partial x} \tag{7}$$

Since $\left\{\frac{d\gamma}{dt}=\frac{\partial\gamma}{\partial t}+\frac{\partial\gamma}{\partial x}.\left(\frac{\partial x}{\partial t}\right)\right\}$

$$\gamma''=\frac{d}{dt}\left(\frac{\partial\gamma}{\partial t}+x'\frac{\partial\gamma}{\partial x}\right)$$

$$\gamma''=\frac{\partial^2\gamma}{\partial t^2}+\frac{\partial}{\partial x}\left(\frac{\partial\gamma}{\partial t}\right).\frac{\partial x}{\partial t}+x'\left\{\frac{\partial}{\partial t}\left(\frac{\partial\gamma}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\gamma}{\partial x}\right).\frac{\partial x}{\partial t}\right\}+\frac{\partial\gamma}{\partial x}.x''$$

$$\gamma''=\frac{\partial^2\gamma}{\partial t^2}+\frac{\partial^2\gamma}{\partial t\partial x}.x'+x'.\frac{\partial^2\gamma}{\partial t\partial x}+x'^2.\frac{\partial^2\gamma}{\partial x^2}+\frac{\partial\gamma}{\partial x}.x''$$

$$\gamma''=\frac{\partial^2\gamma}{\partial t^2}+2x'\frac{\partial^2\gamma}{\partial t\partial x}+x'^2\frac{\partial^2\gamma}{\partial x^2}+\frac{\partial\gamma}{\partial x}x'' \tag{8}$$

and

$$\gamma'''=\frac{\partial^3\gamma}{\partial t^3}+3x'\frac{\partial^3\gamma}{\partial t^2\partial x}+3x''\frac{\partial^2\gamma}{\partial t\partial x}+x'''\frac{\partial\gamma}{\partial x}+3x'^2\frac{\partial^3\gamma}{\partial t\partial x^2}+3x'x''\frac{\partial^2\gamma}{\partial x^2}+x'^3\frac{\partial^3\gamma}{\partial x^3} \tag{9}$$

Similarly, we have

$$\delta'=\frac{\partial\delta}{\partial t}+x'\frac{\partial\delta}{\partial x} \tag{10}$$

Since $\left\{\frac{d\delta}{dt}=\frac{\partial\delta}{\partial t}+\frac{\partial\delta}{\partial x}.\left(\frac{\partial x}{\partial t}\right)\right\}$

$$\delta''=\frac{d}{dt}\left(\frac{\partial\delta}{\partial t}+x'\frac{\partial\delta}{\partial x}\right)$$

$$\delta''=\frac{\partial^2\delta}{\partial t^2}+\frac{\partial}{\partial x}\left(\frac{\partial\delta}{\partial t}\right).\frac{\partial x}{\partial t}+x'\left\{\frac{\partial}{\partial t}\left(\frac{\partial\delta}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\delta}{\partial x}\right).\frac{\partial x}{\partial t}\right\}+\frac{\partial\delta}{\partial x}.x''$$

$$\delta''=\frac{\partial^2\delta}{\partial t^2}+\frac{\partial^2\delta}{\partial t\partial x}.x'+x'.\frac{\partial^2\delta}{\partial t\partial x}+x'^2.\frac{\partial^2\delta}{\partial x^2}+\frac{\partial\delta}{\partial x}.x''$$

$$\delta''=\frac{\partial^2\delta}{\partial t^2}+2x'\frac{\partial^2\delta}{\partial t\partial x}+x'^2\frac{\partial^2\delta}{\partial x^2}+\frac{\partial\delta}{\partial x}x'' \tag{11}$$

and

$$\delta'''=\frac{\partial^3\delta}{\partial t^3}+3x'\frac{\partial^3\delta}{\partial t^2\partial x}+3x''\frac{\partial^2\delta}{\partial t\partial x}+x'''\frac{\partial\delta}{\partial x}+3x'^2\frac{\partial^3\delta}{\partial t\partial x^2}+3x'x''\frac{\partial^2\delta}{\partial x^2}+x'^3\frac{\partial^3\delta}{\partial x^3} \tag{12}$$

Substitute all values of derivatives in equation (6) we get

$$\left[-x'\left(\frac{\partial^3\delta}{\partial t^3}+3x'\frac{\partial^3\delta}{\partial t^2\partial x}+3x''\frac{\partial^2\delta}{\partial t\partial x}+x'''\frac{\partial\delta}{\partial x}+3x'^2\frac{\partial^3\delta}{\partial t\partial x^2}+3x'x''\frac{\partial^2\delta}{\partial x^2}+x'^3\frac{\partial^3\delta}{\partial x^3}\right)-3x''\left(\frac{\partial^2\delta}{\partial t^2}+2x'\frac{\partial^2\delta}{\partial t\partial x}+x'^2\frac{\partial^2\delta}{\partial x^2}+\frac{\partial\delta}{\partial x}x''\right)\right]$$
$$+\left[\frac{\partial^3\gamma}{\partial t^3}+3x'\frac{\partial^3\gamma}{\partial t^2\partial x}+3x''\frac{\partial^2\gamma}{\partial t\partial x}+x'''\frac{\partial\gamma}{\partial x}+3x'^2\frac{\partial^3\gamma}{\partial t\partial x^2}+3x'x''\frac{\partial^2\gamma}{\partial x^2}+x'^3\frac{\partial^3\gamma}{\partial x^3}\right]=0 \tag{13}$$

Using equation (1) in equation (13) we get,

$$\left[-x'\frac{\partial^3\delta}{\partial t^3}-3x'^2\frac{\partial^3\delta}{\partial t^2\partial x}-3x'x''\frac{\partial^2\delta}{\partial t\partial x}-3x'^3\frac{\partial^3\delta}{\partial t\partial x^2}-3x'^2x''\frac{\partial^2\delta}{\partial x^2}-x'^4\frac{\partial^3\delta}{\partial x^3}-3x''\frac{\partial^2\delta}{\partial t^2}-6x'x''\frac{\partial^2\delta}{\partial t\partial x}-3x'^2x''\frac{\partial^2\delta}{\partial x^2}-3(x'')^2\frac{\partial\delta}{\partial x}\right]$$
$$+\left[\frac{\partial^3\gamma}{\partial t^3}+3x'\frac{\partial^3\gamma}{\partial t^2\partial x}+3x''\frac{\partial^2\gamma}{\partial t\partial x}+3x'^2\frac{\partial^3\gamma}{\partial t\partial x^2}+3x'x''\frac{\partial^2\gamma}{\partial x^2}+x'^3\frac{\partial^3\gamma}{\partial x^3}\right]=0 \tag{14}$$

Now we equate the coefficient of various powers of x, x', x'', x''^2 etc.

Coefficient of $\left(x''\right)^2$:
$$-3\frac{\partial\delta}{\partial x}=0$$
$$\delta=a\left(t\right) \tag{15}$$

Coefficient of $\left(x''\right)$:
$$\left(3\frac{\partial^2\gamma}{\partial t\partial x}-3\frac{\partial^2\delta}{\partial t^2}\right)=0$$
$$\frac{\partial^2\gamma}{\partial t\partial x}=\frac{\partial^2\delta}{\partial t^2}$$
$$\frac{\partial^2\gamma}{\partial t\partial x}=a''$$
$$\frac{\partial\gamma}{\partial x}=a'+\mathrm{b}\left(x\right)$$
$$\gamma=a'x+e\left(x\right)+d\left(t\right) \tag{16}$$

Where a and d are functions of t only and e is function of x only.

coefficient of x' :
$$\left(3\frac{\partial^3\gamma}{\partial t^2\partial x}-\frac{\partial^3\delta}{\partial t^3}\right)=0$$
$$3\frac{\partial^2}{\partial t^2}\left(a'+\mathrm{e}'\right)-a'''=0$$
$$3\left(a'''+0\right)-a'''=0$$
$$2a'''=0$$
$$a'''=0$$

After solving we get,
$$a\left(t\right)=h_1\frac{t^2}{2}+h_2t+h_3 \tag{17}$$

Where h_1,h_2 and h_3 are arbitrary constants.

Coefficients of $x'x''$:
$$\left(3\frac{\partial^2\gamma}{\partial x^2}-9\frac{\partial^2\delta}{\partial t\partial x}\right)=0$$
$$\left(3\frac{\partial^2\gamma}{\partial x^2}-0\right)=0$$
$$\frac{\partial^2\gamma}{\partial x^2}=0$$
$$\frac{\partial}{\partial x}\left(a'+e'\right)=0$$
$$e''=0$$

After solving, we get
$$e\left(x\right)=h_4x+h_5 \tag{18}$$

Where h_4 and h_5 are arbitrary constants.

Coefficients of constants:
$$\frac{\partial^3\gamma}{\partial t^3}=0$$
$$\left(a''''x+d'''\right)=0$$

Equate constants, we have,
$$d'''=0$$

After solving, we get,

$$d(t) = h_6 \frac{t^2}{2} + h_7 t + h_8 \tag{19}$$

Where h_6, h_7 and h_8 are arbitrary constants.

By equation (15) we have,

$$\delta = h_1 \frac{t^2}{2} + h_2 t + h_3 \tag{20}$$

Putting all the values from $(17),(18),(19)$ in equation (16) we get

$$\gamma = \left(h_1 \frac{t^2}{2} + h_2 t + h_3 \right)' x + \left(h_6 \frac{t^2}{2} + h_7 t + h_8 \right) + (h_4 x + h_5)$$

$$\gamma = (h_1 t + h_2) x + \left(h_6 \frac{t^2}{2} + h_7 t + h_8 \right) + (h_4 x + h_5)$$

$$\gamma = (h_1 t + h_2) x + \left(h_6 \frac{t^2}{2} + h_7 t + h_9 + h_4 x \right)$$

$$\gamma = (h_1 t x + (h_2 + h_4) x) + \left(h_6 \frac{t^2}{2} + h_7 t + h_9 \right)$$

$$\gamma = (h_1 t x + h_{10} x) + \left(h_6 \frac{t^2}{2} + h_7 t + h_9 \right) \tag{21}$$

Where $h_9 = h_8 + h_5$ and $h_{10} = h_2 + h_4$ are arbitrary constants.

The Generator S of the infinitesimal transformation is of the form,

$$S = \delta \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}$$

$$S = \left(h_1 \frac{t^2}{2} + h_2 t + h_3 \right) \frac{\partial}{\partial t} + \left(h_1 t x + h_{10} x + h_6 \frac{t^2}{2} + h_7 t + h_9 \right) \frac{\partial}{\partial x} \tag{22}$$

Which is seven parameter symmetry are given as

$$S^1 = \frac{t^2}{2} \frac{\partial}{\partial t} + t x \frac{\partial}{\partial x}$$

$$S^2 = t \frac{\partial}{\partial t}$$

$$S^3 = \frac{\partial}{\partial t}$$

$$S^4 = x \frac{\partial}{\partial x}$$

$$S^5 = \frac{\partial}{\partial x}$$

$$S^6 = \frac{t^2}{2} \frac{\partial}{\partial x}$$

$$S^7 = t \frac{\partial}{\partial x} \tag{23}$$

These all are infinitesimal generators for equation (1)

Now, we take

$$S^7 = t\frac{\partial}{\partial x}$$

Which are the Lie solvable algebra of admitted seven one parameter symmetry (23)

Third order prolongation is obtained as follows

$$S^7{}_{[0]} = t\frac{\partial}{\partial x} + 0.\frac{\partial}{\partial t} = t\frac{\partial}{\partial x}$$

$$S^7{}_{[1]} = S^7{}_{[0]} + \left(-x'.0+1\right)\frac{\partial}{\partial x'} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \tag{24}$$

$$S^7{}_{[2]} = S^7{}_{[1]} + \left(-x'.0 - 2x''.0 + 0\right)\frac{\partial}{\partial x''}$$

$$S^7{}_{[2]} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} + 0\frac{\partial}{\partial x''} \tag{25}$$

$$S^7{}_{[3]} = S^7{}_{[2]} + \left(-x'.0 - 3x''.0 - 3x'''.0 + 0\right)\frac{\partial}{\partial x'''}$$

$$S^7{}_{[3]} = t\frac{\partial}{\partial x} + 0\frac{\partial}{\partial t} + \frac{\partial}{\partial x'} + 0\frac{\partial}{\partial x''} + 0\frac{\partial}{\partial x'''} \tag{26}$$

Now, we must solve

$$\frac{dx}{t} = \frac{dt}{0} = \frac{dx'}{1} = \frac{dx''}{0} = \frac{dx'''}{0} \tag{27}$$

Case 1[st] :
$$\frac{dx}{t} = \frac{dt}{0}$$

$$t = u \tag{28}$$

Where u is a constant.

Case 2[nd] :
$$\frac{dx}{t} = \frac{dx'}{1}$$

Integrating on both sides, we get

$$\int\frac{dx}{t} = \int\frac{dx'}{1}$$

$$\frac{x}{t} = x' + v_1$$

$$v_1 = \frac{x}{t} - x' \tag{29}$$

Where v_1 is a constant.

Case 3[rd] :
$$\frac{dx'}{1} = \frac{dx''}{0}$$

On solving, we get

$$x'' = v \tag{30}$$

Where v is a constant.

Equation (1) can be reduced as follows

$$\frac{dv}{du} = \frac{D_t(v)}{D_t(u)}$$

$$\frac{dv}{du} = \frac{x'''}{1} \tag{31}$$

Using equation (1) in equation (31) then

$$\frac{dv}{du} = 0$$
$$dv = 0$$
$$v = k$$
$$x'' = k$$
$$x' = kt + k_1$$
$$x = k\frac{t^2}{2} + k_1 t + k_2 \tag{32}$$

Where k, k_1, k_2 are arbitrary constants.

Equation (32) gives us the solution of equation (1) with the use of Lie symmetry method.

IV. CONCLUSION

In this research paper, we solved third order linear ordinary differential equations with the help of Lie symmetry method. In the order of solution of linear ODE, firstly we made third extension and made the determining equations. After solving determining equations, we found infinitesimal generators for ODEs. we took one of the infinitesimal generator which can be solvable easily. Now, the order of ODE has been reduced into two which is solvable very easily.

Lie symmetry method is one of the powerful tool for solving differential equations. Lie symmetry method can be apply in any of the differential equation. It reduces the order of differential equation as we have discussed in this paper. It gives the exact solution of differential equations.

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