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# Existence of Solutions for Boundary Value Problems of Fractional Pantograph Integrodi erential Equations with Multi-Fractional Derivatives

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***Abstract***

This study investigates the existence of solutions to boundary value problems(BVPs) using fractional pantograph integrodifferential equations with several fractional derivatives (Caputo and Riemann-Liouville). The existence of solutions is established using Krasnoselskii's fixed-point theorem, while uniqueness is verified using the Banach contraction principle. An example is offered to demonstrate the theoretical results' validity.

**Keywords:** Fractional calculus, Existence results, Boundary value problem.

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## I. Introduction

Fractional differential equations have recently gained prominence in advancing special functions and integral transforms. Their applications extend across various fields, including biology, control theory, bioengineering, biomedical sciences, economics, and variational problems, among others. For additional insights, please consult references [3, 4, 12, 13] and the related literature.

As a result of these advancements, the question of existence for solutions to fractional differential equations within these models has captured the interest of numerous mathematical researchers. This has led to a proliferation of articles discussing the existence of solutions for initial, boundary, and nonlocal fractional equations, employing various types of fractional derivatives (see references [10, 11] for further exploration). Significant contributions addressing both the integral operator and arbitrary fractional-order differential operators can be found in [5, 6, 10]. The main focus of this field of study is on creating theoretical models, instruments, and procedures for the examination and resolution of fractional differential equations (FDEs). Researchers can develop effective methods for solving FDEs and learn more about their practical applications by studying the behavior and characteristics of their solutions. In recent years, many authors have looked at the existence and uniqueness theorems for FDEs with mixed fractional derivatives [1, 2, 7]. Wang [15] explored the stability outcomes of neutral fractional functional differential equations with several Caputo fractional derivatives, applying conventional fixed point theorems. Numerous researchers have studied multifractional derivatives, enhancing their application in fields such as control theory and biological systems. These derivatives enable accurate modeling of dynamic systems by incorporating memory effects and historical influences [8, 9, 14, 15] Building on the examination and findings related to the discussed issues, this research investigates the existence and uniqueness of solutions to the following nonlinear fractional differential equations boundary value problem using multi-fractional derivatives:

$$\varepsilon\, {}^{C}D^{\gamma}u(t) - {}^{RL}D^{\alpha}u(t) = \mathfrak{f}\left(t, u(t), u(\lambda t), \int_0^t \mathfrak{g}(t, s, u(s))ds\right), \quad t \in [0,1] = J, \tag{1.1}$$

$$u(0) = 0, \quad u'(1) = 0, \varepsilon > 0. \tag{1.2}$$

Here, ${}^{C}D^{\gamma}$ represents the Caputo derivative (CD) of fractional order $\gamma$ with $1 < \gamma < 2$, while ${}^{\mathrm{RL}}D^{\alpha}$ denotes the Riemann-Liouville(RL) derivative of fractional order $\alpha$ with $0 < \alpha < 1$ and $\lambda \in (0, 1)$. Let $\mathfrak{f} : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathfrak{g} : J \times J \times \mathbb{R} \to \mathbb{R}$ be continuous functions.

- Problem and Approach:
  We examine boundary value problems for fractional pantograph integrodifferential equations involving multi-fractional derivatives, specifically Caputo and Riemann-Liouville.

  Action: Introduce the specific problem and the types of fractional derivatives being considered.

- Existence and Uniqueness:
  The existence of solutions is established using Krasnoselskii's fixed point theorem, while the uniqueness is proven via the Banach contraction mapping principle.

  Action: Clearly state the methods used to demonstrate both the existence and uniqueness of the solutions.

- Example for Validation:
  An example is presented to verify the theoretical results and illustrate their practical relevance.

  Action: Mention the inclusion of an example to validate the theoretical findings and demonstrate their practical applicability.

The remainder of this paper is structured as follows: Section 2 provides the essential definitions and the fundamental tools that will be utilized in the following sections. Section 3 establishes and solves the different conditions for the existence and uniqueness of solutions for multi-fractional derivatives. Finally, a specific example is included to illustrate the results obtained

## 2. Preliminaries

In this section, we introduce essential definitions, lemmas, and theorems required to establish the main results.

**Definition 2.1.** *The RL fractional integral of order $\gamma > 0$ for a function $\mathfrak{f} : [a, b] \to \mathbb{R}$ evaluated at a point $t$ is given by the following definition:*

$$I_a^{\gamma}\mathfrak{f}(t) = \int_a^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{f}(\tau)\, d\tau,$$

*where $\Gamma$ represents the Gamma function, assuming the right-hand side is defined at each point.*

**Definition 2.2.** *The RL fractional derivative of order $\gamma > 0$ for a function $\mathfrak{f} : [a, b] \to \mathbb{R}$ at a point $t$ is defined by:*

$$D_a^{\gamma}\mathfrak{f}(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t \frac{(t-\tau)^{n-\gamma-1}}{\Gamma(n-\gamma)}\mathfrak{f}(\tau)\, d\tau,$$

*where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of $\gamma$, assuming the right-hand side is point-wise defined.*

**Definition 2.3.** *The Caputo derivative of fractional order $\gamma$ for a function $\mathfrak{f} : [a, b] \to \mathbb{R}$ that is $n$-times differentiable is defined as:*

$$^{C}D_{a}^{\gamma}\mathfrak{f}(t) = \frac{1}{\Gamma(n-\gamma)} \int_{a}^{t} (t-\tau)^{n-\gamma-1} \left(\frac{d}{d\tau}\right)^{n} \mathfrak{f}(\tau)\, d\tau,$$

*where $n = [\gamma] + 1$ and $\gamma > 0$.*

**Remark 2.4.** *Assume $0 < \gamma \leqslant 1$ and $m = [\gamma] + 1$. If $x(t) \in C^{m}[0, 1]$, then*

$$^{C}I_{a}^{\gamma}\, {}^{C}D_{a}^{\gamma}x(t) = x(t) - x(0).$$

**Theorem 2.5.** *(Krasnoselskii's Fixed Point Theorem): Let $M$ be a closed, convex, and bounded nonempty subset of a Banach space $X$. Consider two operators $P$ and $Q$ that satisfy the following criteria:*

- *$Px + Qy \in M$ for all $x, y \in M$;*
- *$Q$ is a contraction mapping;*
- *$P$ is both compact and continuous.*

*Then, there exists an element $\gamma \in M$ such that the equation $\gamma = P\gamma + Q\gamma$ is satisfied.*

**Theorem 2.6.** *(Contraction Mapping Principle): Let $M$ be a Banach space. If $T : M \to M$ is a contraction, then $T$ possesses a unique fixed point in $M$.*

## 3. Main Results

This section will demonstrate the existence and uniqueness of a solution for problem (1.1) – (1.2) in the Banach space $C$ by utilizing the Banach contraction principle and the Krasnoselskii fixed point theorem. We will first examine the fundamental assumptions required for the forthcoming analysis:

(A1) Let $\mathfrak{f}(C([0, 1], \mathbb{R}))$ represent the Banach space consisting of all continuous functions that map the interval $J$ to $\mathbb{R}$, endowed with the norm defined by

$$\|u\| = \sup\{|u(t)| : t \in J\}.$$

(A2) There exist constants $\mathscr{L}, \mathscr{L}_1, \mathscr{L}_2 > 0$ such that

$$|\mathfrak{f}(t, x_1, x_2, x_3) - \mathfrak{f}(t, x_1^*, x_2^*, x_3^*)| \leqslant \mathscr{L}_1(|x_1 - x_1^*| + |x_2 - x_2^*| + |x_3 - x_3^*|),$$
$$|\mathfrak{g}(t, s, x_1) - \mathfrak{g}(t, s, x_1^*)| \leqslant \mathscr{L}_2(|x_1 - x_1^*|),$$
$$|\mathfrak{f}(t, x_1, x_2, x_3)| \leqslant \mathscr{L}.$$

for all $x_i, x_i^* \in \mathbb{R}$, for $i = 1, 2, 3$, and $t \in J$.

(A3) For each $\eta^* > 0$, let $B_\eta^*$ be defined as the set $B_{\eta^*} = \{u \in C(J, \mathbb{R}), \|u\| \leqslant \eta^*\}$. It is evident that $B_{\eta^*}$ is a bounded, closed, and convex subset of $C(J, \mathbb{R})$.

**Lemma 1.** *The solution to the BVPs (1.1) – (1.2) satisfies the integral equation as follows:*

$$\begin{aligned} u(t) =& \frac{1}{\varepsilon}I^{\gamma-\alpha}u(t) + \frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t, u(t), u(\lambda t), \int_0^t \mathfrak{g}(t, s, u(s))ds\right) - \frac{1}{\varepsilon}I^{\gamma-\alpha-1}u(1) \\ &- \frac{1}{\varepsilon}I^{\gamma-1}\mathfrak{f}\left(1, u(1), u(\lambda 1), \int_0^1 \mathfrak{g}(1, s, u(s))ds\right). \end{aligned} \tag{3.1}$$

*Proof.* From Equation (1.1), we have

$$^{C}D^{\gamma}u(t) = \frac{1}{\varepsilon}{}^{RL}D^{\alpha}u(t) + \frac{1}{\varepsilon}\mathfrak{f}\left(t, u(t), u(\lambda t), \int_0^t \mathfrak{g}(t, s, u(s))ds\right).$$

By applying the RL fractional integral of order $\gamma$ to both sides, we obtain

$$\begin{aligned} u(t)-u(0)=&\frac{1}{\varepsilon}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)+a_1+a_2u\\ u(t)=&\frac{1}{\varepsilon}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)+a_1+a_2u. \end{aligned} \tag{3.2}$$

Thus, the condition $u(0)=0$ implies that $a_1=0$. Then,

$$u(t)=\frac{1}{\varepsilon}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)+a_1+a_2u.$$

By differentiating both sides, we obtain:

$$u'(t)=\frac{1}{\varepsilon}\frac{d}{dt}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}\frac{d}{dt}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)+a_2.$$

Using the boundary condition $u'(1)=0$ in the equation mentioned earlier, we derive

$$a_2=-\frac{1}{\varepsilon}I^{\gamma-\alpha-1}u(1)-\frac{1}{\varepsilon}I^{\gamma-1}\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right).$$

By inserting the values of $a_1$ and $a_2$ into Equation (3.2), we derive:

$$\begin{aligned} u(t)=&\frac{1}{\varepsilon}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)-\frac{1}{\varepsilon}I^{\gamma-\alpha-1}u(1)\\ &-\frac{1}{\varepsilon}I^{\gamma-1}\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right). \end{aligned}$$ □

Next, we shall prove the existence of a solution to the problem by applying Krasnoselskii's fixed point theorem.

**Theorem 3.1.** *Suppose that conditions* $(A1)-(A3)$ *hold, and if* $\mathcal{N}_1<1$, *then the BVPs* (1.1) − (1.2) *possesses at least one solution in* $C(J,\mathbb{R})$.

*Proof.* In order to demonstrate that the problem (1.1) − (1.2) has a solution, we are going to illustrate the proof by taking into consideration the subsequent steps:
**Step 1:** We shall define two operators $\mathscr{A}_1$ and $\mathscr{A}_2$, as follows for any constant $\varepsilon>0$:

$$\mathscr{A}_1=\frac{1}{\varepsilon}\,I^{\gamma-\alpha}u(t) \tag{3.3}$$

$$\begin{aligned} \mathscr{A}_2=&\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)-\frac{1}{\varepsilon}I^{\gamma-1}\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right)\\ &-\frac{1}{\varepsilon}I^{\gamma-\alpha-1}u(1). \end{aligned} \tag{3.4}$$

Next, we will demonstrate that the operator $\mathscr{A}_1+\mathscr{A}_2=\mathscr{A}$ is bounded as follows:

$$\begin{aligned} |\mathscr{A}u(t)|\leqslant&\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(\tau)|d\tau\\ &+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(\tau,u(\tau),u(\lambda\tau),\int_0^\tau\mathfrak{g}(\tau,s,u(s))ds\right)\right|d\tau\\ &+\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(1)|d\tau\\ &+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right)\right|d\tau,\\ \leqslant&\frac{\eta^*}{\varepsilon\,\Gamma(\gamma-\alpha-1)}+\frac{\mathscr{L}}{\varepsilon\,\Gamma(\gamma)}+\frac{|x(1)|}{\Gamma(\gamma-\alpha)}+\frac{\left|\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right)\right|}{\Gamma(\gamma)}. \end{aligned}$$

By taking the supremum on both sides, we get:

$$\|\mathscr{A}u\| \leqslant \frac{\mathscr{M}}{\varepsilon},$$

where $\mathscr{M} = \dfrac{\eta^*}{\Gamma(\gamma-\alpha-1)} + \dfrac{\mathscr{L}}{\Gamma(\gamma)} + \dfrac{|x(1)|}{\Gamma(\gamma-\alpha)} + \dfrac{\left|\mathfrak{f}\left(1, u(1), u(\lambda 1), \int_0^1 \mathfrak{g}(1,s,u(s))ds\right)\right|}{\Gamma(\gamma)}$. Therefore, the operator $\mathscr{A}$ is bounded.

**Step 2:** Contraction.

$$\begin{aligned}
|\mathscr{A}_1 u(t) - \mathscr{A}_1 v(t)| \leqslant & \left|\frac{1}{\varepsilon} I^{\gamma-\alpha} u(t) - \frac{1}{\varepsilon} I^{\gamma-\alpha} v(t)\right| \\
\leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u(\tau)| d\tau - \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |v(\tau)| d\tau \\
\leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u(\tau) - v(\tau)| d\tau.
\end{aligned}$$

Taking the supremum on both sides yields:

$$\begin{aligned}
\|\mathscr{A}_1 u - \mathscr{A}_1 v\| \leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \|u-v\| d\tau \\
\leqslant & \frac{\|u-v\|}{\varepsilon\Gamma(\gamma-\alpha+1)} \\
\leqslant & \mathcal{N}_1 \|u-v\|,
\end{aligned}$$

where $\mathcal{N}_1 = \dfrac{1}{\varepsilon\Gamma(\gamma-\alpha+1)}$. Therefore, the operator $\mathscr{A}_1$ is a contraction.

**Step 3:** To prove the complete continuity of the operator $\mathscr{A}_2$, it is crucial to show both its continuity and equicontinuity. By establishing these two properties, we can

$$\begin{aligned}
|\mathscr{A}_2 u_n(t) - \mathscr{A}_2 u(t)| \leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u_n(\tau) - u(\tau)| d\tau \\
& + \frac{1}{\varepsilon\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \Bigg| \mathfrak{f}\left(\tau, u_n(\tau), u_n(\lambda\tau), \int_0^\tau \mathfrak{g}(\tau,s,u_n(s))ds\right) \\
& - \mathfrak{f}\left(\tau, u(\tau), u(\lambda\tau), \int_0^\tau \mathfrak{g}(\tau,s,u(s))ds\right) \Bigg| d\tau \\
& + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u_n(1) - u(1)| d\tau \\
& + \frac{1}{\varepsilon\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \Bigg| \mathfrak{f}\left(1, u_n(1), u_n(\lambda 1), \int_0^1 \mathfrak{g}(1,s,u_n(s))ds\right) \\
& - \mathfrak{f}\left(1, u(1), u(\lambda 1), \int_0^1 \mathfrak{g}(1,s,u(s))ds\right) \Bigg| d\tau \\
\leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u_n(\tau) - u(\tau)| d\tau \\
& + \frac{1}{\varepsilon\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \mathscr{L}_1 (|u_n(\tau) - u(\tau)| + |u_n(\lambda\tau) - u(\lambda\tau)| \\
& + \mathscr{L}_2 |u_n(\tau) - u(\tau)|) d\tau \\
& + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} |u_n(1) - u(1)| d\tau
\end{aligned}$$

$$+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(|u_n(1)-u(1)|+|u_n(\lambda 1)-u(\lambda 1)| + \mathscr{L}_2|u_n(\tau)-u(\tau)|)d\tau.$$

Taking the supremum on both sides yields:

$$\begin{aligned}\|\mathscr{A}_2 u_n - \mathscr{A}_2 u\| \leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t (t-\tau)^{\gamma-\alpha-1}\|u_n-u\|d\tau \\ & +\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(\|u_n-u\|+\|u_n-u\|+\mathscr{L}_2\|u_n-u\|)d\tau \\ & +\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t (t-\tau)^{\gamma-\alpha-1}\|u_n-u\|d\tau \\ & +\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(\|u_n-u\|+\|u_n-u\|+\mathscr{L}_2\|u_n-u\|)d\tau \\ \leqslant & \frac{1}{\varepsilon}\left[\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(\gamma-\alpha)}+\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma)}\right]\|u_n-u\|.\end{aligned}$$

As $u_n \to u$ as $n \to \infty$. Hence, $\mathscr{A}_2$ is continuous.

**Step 4:** We shall then prove that $\mathscr{A}_2$ is an equicontinuous operator.

$$\begin{aligned}|\mathscr{A}u(t_2)-\mathscr{A}u(t_1)| \leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_2}(t_2-\tau)^{\gamma-\alpha-1}|u(\tau)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma)}\int_0^{t_2}(t_2-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(\tau,u(\tau),u(\lambda\tau),\int_0^{\tau}\mathfrak{g}(\tau,s,u(s))ds\right)\right|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_2}(t_2-\tau)^{\gamma-\alpha-1}|u_n(1)-u(1)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma)}\int_0^{t_2}(t_2-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(1,u_n(1),u_n(\lambda 1),\int_0^{1}\mathfrak{g}(1,s,u_n(s))ds\right)\right|d\tau \\ & - \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_1}(t_1-\tau)^{\gamma-\alpha-1}|u(\tau)|d\tau \\ & - \frac{1}{\varepsilon\Gamma(\gamma)}\int_0^{t_1}(t_1-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(\tau,u(\tau),u(\lambda\tau),\int_0^{\tau}\mathfrak{g}(\tau,s,u(s))ds\right)d\tau\right| \\ & - \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_1}(t_1-\tau)^{\gamma-\alpha-1}|u_n(1)-u(1)|d\tau \\ & - \frac{1}{\varepsilon\Gamma(\gamma)}\int_0^{t_1}(t_1-\tau)^{\gamma-\alpha-1}\left|\mathfrak{f}\left(1,u_n(1),u_n(\lambda 1),\int_0^{1}\mathfrak{g}(1,s,u_n(s))ds\right)\right|d\tau \\ \leqslant & \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_1}((t_2-\tau)^{\gamma-\alpha-1}-(t_1-\tau)^{\gamma-\alpha-1})|u(\tau)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\gamma-\alpha-1}|u(\tau)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma)}\int_{t_1}^{t_2}((t_2-\tau)^{\gamma-\alpha-1})\left|\mathfrak{f}\left(\tau,u(\tau),u(\lambda\tau),\int_0^{\tau}\mathfrak{g}(\tau,s,u(s))ds\right)\right|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^{t_1}((t_2-\tau)^{\gamma-\alpha-1}-(t_1-\tau)^{\gamma-\alpha-1})|u_n(1)-u(1)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_{t_1}^{t_2}((t_2-\tau)^{\gamma-\alpha-1})|u_n(1)-u(1)|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma)}\int_0^{t_1}((t_2-\tau)^{\gamma-\alpha-1}-(t_1-\tau)^{\gamma-\alpha-1})\left|\mathfrak{f}\left(1,u_n(1),u_n(\lambda 1),\int_0^{1}\mathfrak{g}(1,s,u_n(s))ds\right)\right|d\tau \\ & + \frac{1}{\varepsilon\Gamma(\gamma)}\int_{t_1}^{t_2}((t_2-\tau)^{\gamma-\alpha-1})\left|\mathfrak{f}\left(1,u_n(1),u_n(\lambda 1),\int_0^{1}\mathfrak{g}(1,s,u_n(s))ds\right)\right|d\tau. \qquad \square\end{aligned}$$

As $t_2$ approaches $t_1$, the right-hand side of the inequality above converges to zero. Therefore, the operator $\mathscr{A}_2$ is equicontinuous. Since all the conditions of Krasnoselskii's fixed point theorem are fulfilled, especially the existence of a fixed point, this concludes the proof.

**Theorem 3.2.** *Assume that (A2) is hold. If* $\frac{1}{\varepsilon}\left[\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(\gamma-\alpha)}+\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma)}\right] < 1$, *then, problem* (1.1) − (1.2), *has a unique solution.*

*Proof.* First, we define the operator $T$.

$$\begin{aligned}Tu(t) =&\frac{1}{\varepsilon}I^{\gamma-\alpha}u(t)+\frac{1}{\varepsilon}I^{\gamma}\mathfrak{f}\left(t,u(t),u(\lambda t),\int_0^t\mathfrak{g}(t,s,u(s))ds\right)-\frac{1}{\varepsilon}I^{\gamma-\alpha-1}u(1)\\&-\frac{1}{\varepsilon}I^{\gamma-1}\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right).\end{aligned}$$

Next, we will employ the Banach contraction mapping theorem to establish the uniqueness of $T$.

$$\begin{aligned}|Tu(t)-Tv(t)|\leqslant&\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(\tau)-v(\tau)|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\Bigg|\mathfrak{f}\left(\tau,u(\tau),u(\lambda\tau),\int_0^\tau\mathfrak{g}(\tau,s,u(s))ds\right)\\&-\mathfrak{f}\left(\tau,v(\tau),v(\lambda\tau),\int_0^\tau\mathfrak{g}(\tau,s,v(s))ds\right)\Bigg|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(1)-v(1)|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\Bigg|\mathfrak{f}\left(1,u(1),u(\lambda 1),\int_0^1\mathfrak{g}(1,s,u(s))ds\right)\\&-\mathfrak{f}\left(1,v(1),v(\lambda 1),\int_0^1\mathfrak{g}(1,s,v(s))ds\right)\Bigg|d\tau\\&\leqslant\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(\tau)-v(\tau)|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(|u(\tau)-v(\tau)|+|u(\lambda\tau)-v(\lambda\tau)|\\&+\mathscr{L}_2|u_(\tau)-v(\tau)|)d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}|u(1)-v(1)|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(|u(1)-v(1)|+|u(\lambda 1)-v(\lambda 1)|\\&+\mathscr{L}_2|u(\tau)-v(\tau)|)d\tau.\end{aligned}$$

Taking the supremum on both sides yields:

$$\begin{aligned}\|Tu-Tv\|\leqslant&\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\|u-v\|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(\|u-v\|+\|u-v\|+\mathscr{L}_2\|u-v\|)d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma-\alpha)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\|u-v\|d\tau\\&+\frac{1}{\varepsilon\Gamma(\gamma)}\int_0^t(t-\tau)^{\gamma-\alpha-1}\mathscr{L}_1(\|u-v\|+\|u-v\|+\mathscr{L}_2\|u-v\|)d\tau\\&\leqslant\frac{1}{\varepsilon}\left[\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(\gamma-\alpha)}+\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma)}\right]\|u-v\|.\end{aligned}$$

□

Upon examining the circumstances, it is evident that the conditions of the Banach contraction principle are met, allowing us to deduce that a unique fixed point exists for the specified problem.

## 4. Example

Consider the problem

$$\frac{11}{100} {}^{C}D^{3/2}u(t) - {}^{RL}D^{2/3}u(t) = \frac{\cos u(t)}{5}\frac{e^t}{1+e^t} + u(\lambda t)\frac{e^t}{1+e^t} + \int_0^t \frac{e^s}{1+e^s}u(s)ds, \quad t \in [0,1], \tag{4.1}$$

$$u(0) = 0, \quad u'(1) = 0. \tag{4.2}$$

Here

$$\alpha = \frac{2}{3}, \gamma = \frac{3}{2}, \quad \varepsilon = \frac{11}{100},$$

$$\mathfrak{f}\left(t, u(t), u(\lambda t), \int_0^t \mathfrak{g}(t,s,u(s))ds\right) = \frac{\cos u(t)}{5}\frac{e^t}{1+e^t} + u(\lambda t)\frac{e^t}{1+e^t} + \int_0^t \frac{e^s}{1+e^s}u(s)ds.$$

It is clear that

$$|\mathfrak{f}(t,x_1,x_2,x_3) - \mathfrak{f}(t,x_1^*,x_2^*,x_3^*)| \leqslant \frac{e^t}{1+e^t}(|x_1 - x_1^*| + |x_2 - x_2^*| + |x_3 - x_3^*|),$$

$$|\mathfrak{g}(t,s,x_1) - \mathfrak{g}(t,s,x_1^*)| \leqslant \frac{e^t}{1+e^t}(|x_1 - x_1^*|),$$

for all $x_i, x_i^* \in \mathbb{R}$, for $i = 1,2,3$, and $t \in J$, which satisfies condition (A2). Here $\mathscr{L} = \frac{e^t}{1+e^t} < 1$. Therefore, by applying the concept of uniqueness and using the Lipschitz condition,

$$\frac{1}{\varepsilon}\left[\frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma+1)} + \frac{1}{\Gamma(\gamma-\alpha)} + \frac{\mathscr{L}_1(2+\mathscr{L}_1)}{\Gamma(\gamma)}\right] < 0.5448 < 1.$$

Based on our analysis, we can infer that the boundary value problem (4.1) – (4.2) has a unique solution.

## 5. Conclusion

This research study delves into the analysis of nonlinear multifractional differential equations, with a particular emphasis on mixed fractional differential equation BVPs. To demonstrate existence findings, the study uses Krasnoselskii's fixed point theorem, while a uniqueness theorem is derived using the Banach contraction mapping principle. Furthermore, the use of an exemplary example confirms the validity of the acquired results.

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