International Journal Of Engineering Research And Development
e- ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 20, Issue 11 (November, 2024), PP 600-607

Study of shear behavior of a cylindrical tube in great or infinitesimal transformation: Application to three forms of models

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***Abstract**:*
In this research work we have proposed to study a hollow cylindrical tubular structure subjected to an anti-plane shear. we used three energy functions of deformation which is polynomial, power and exponential form respectively to determine the shear solution. The calculations allowed us with certain conditions to find most of the time logarithmic solution of the shear excepted onetime where we find a power solution with a logarithmic term. The numerical simulation of the anti-plane shear shows that in finitesimal transformation, the Diouf-Zidi model give the greatest shear followed by the Knowles-Sternberg shearwhile the Delfino model records the lowest shear, but in great transformation, i.e when the radius is greater than five meters, the Knowles-Sternberg model isbiger than the Diouf-Zidi model with that of Delfino wich can be biger, between or smaller than these two models according to the value of its derivative. Study shows that the form of a model influences the shear only in great transformation, in infinitesimal transformation, shear solutions are all equivalent.

***Keywords**: Anti-planar shear, Incompressible deformation, Infinitesimal or great transformation, Isotropic energy function, Elementary invariants, Cauchy stress tensor.*

Date of Submission: 09-11-2024 Date of Acceptance: 21-11-2024

I. Introduction

The study of shearing of elastic and incompressible materials has always been the subject of special attention in the study of mechanical systems [1]. In the study of mechanical fracture, the example of anti-planar shear has been of particular interest to better understand these mechanical systems. Simple shear deformations, for which the displacement gradient is constant, are sustainable both in the linear and nonlinear theory. So that necessary and sufficient conditions on the strain energies for homogeneous isotropic nonlinear elastic materials which do allow antiplane shear were obtained in Knowles for further contributions in the compressible case [2].

In the linear transversely isotropic elasticity, a study of the deformation of a circular hollow cylinder, whose inner surface is fixed, while its outer surface is subject to a constant axial surface traction is done [3]. In isotropic linear elasticity, the solution of this problem is just a state of anti-plane axial shear. The autors show that it is possible to use an axial tension field to generate an azimuthal shear deformation. they show that this fact suggests to use anisotropyto design some elastic machines which can combine different deformation modes. Other authors [4] have shown that this characterization of materials is closely related to the nature and form of the energy function. This characterization remains less obvious in nonlinear elasticity.

Other studies have focused on the effect of shear stress in general by a fluid ina tubular structure [5]. Their study showed that in the renal tube reduced fluid shear stress down-regulated the levels of megalin receptors, thereby reducingthe renal distribution of albumin nanoparticles.

To describe the anisotropic hyperelastic mechanical behavior of a mechanical structure, it is still useful to use deformation energy functions in form polynomial, exponential, power or logarithmic. These energy potentials have been established as part of a phenomenological approach that describes the macro-scopic nature of the material.

The study of the anti-plane shear of a cylindrical tubular structure in the case of a great or infinitesimal transformation with a three-way application of energy functions will be our contribution in the biomechanical modeling. After the calculation of the anti planar-shear of the three models of our study, these founded shears will be simulated and analyzed with some boundary conditions which are given on the parameters according to whether that the radius increases in great and infinitesimal transformation.

II. Formulation of the problem

Let's consider a continuous material body. the whole of the particles of this body occupies, every moment, an open and connected domain or connected by arc of the physical space. The geometric domain is a hollow cylinder composed of an elastic, isotropic material with an inner surface bounded by a rigid cylinder and an outer surface subjected to axial shear. In a cylindrical coordinate system, let's consider a point *M* which, in the undeformed configuration has the components (R, Θ, Z) and the deformed configuration (r, θ, x). The kinematicsof deformation is described in [6] by:

$$r = r(R); \qquad \theta = \Theta; \qquad z = Z + \omega(R), \tag{1}$$

which translates for axial shear, a combined deformation of the tube: radial with $r(R)$ and longitudinal or anti-plan shear with $\omega(R)$.
With clearly defined boundary conditions on the inner R_i and outer R_e radius [7]. According to (1), we find the following deformation gradient tensor:

$$\mathbf{F} = \begin{pmatrix} r' & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ \omega' & 0 & 1 \end{pmatrix} \tag{2}$$

where r' and ω' are respectively the derivatives with respect to R of r and ω.
From the deformation gradient, we can calculate:
The right Cauchy-Green tensor in the case of a representation in Lagrangian configuration which is defined by:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} r'^2 + \omega'^2 & 0 & \omega' \\ 0 & \frac{r^2}{R^2} & 0 \\ \omega' & 0 & 1 \end{pmatrix} \tag{3}$$

But also the left Cauchy-Green tensor in the case of a representation in Eulerian configuration defined by:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \begin{pmatrix} r'^2 & 0 & r'\omega' \\ 0 & \frac{r^2}{R^2} & 0 \\ r'\omega' & 0 & 1 + \omega'^2 \end{pmatrix} \tag{4}$$

It should be noted that these two representation are equivalent and they are confused in the case of an infinitesimal transformation.
It's follow the first three elementary invariants of $\mathbf{C}$ or $\mathbf{B}$ given by:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}) = r'^2 + \left(\frac{r}{R}\right)^2 + \omega'^2 + 1; \\ I_2 &= tr(\mathbf{C}^*) = tr(\mathbf{B}^*) = r'^2 + \left(\frac{r}{R}\right)^2 \left(1 + \omega'^2\right) + \left(\frac{r}{R}\right)^2; \\ I_3 &= det(\mathbf{C}) = det(\mathbf{B}) = \left(\frac{rr'}{R}\right)^2. \end{aligned} \tag{5}$$

Where tr defines the *trace* operator, det the *determinant* operator and $\mathbf{C}^*$ and $\mathbf{B}^*$ are respectively the adjoints of the tensors $\mathbf{C}$ and $\mathbf{B}$ which are defined by:

$$\mathbf{C}^* = det(\mathbf{C})\mathbf{C}^{-1}; \qquad \mathbf{B}^* = det(\mathbf{B})\mathbf{B}^{-1}. \tag{6}$$

It should be noted also that the elementary invariants allows us to obtain the energy potential W also called the deformation energy function which is a function of these invariants $(W(I_1, I_2, I_3))$. This function translates the mechanical and/or thermodynamic behavior of the material.
To translate the reaction of the material when it is submited to stresses which it undergoes, we introduce a tensor $\boldsymbol{\sigma}$ called Cauchy stress tensor.
This Cauchy stress tensor is given in [8]:

$$\boldsymbol{\sigma} = \beta_0 \mathbf{1} + \beta_\varepsilon \mathbf{C} + \beta_{-\varepsilon} \mathbf{C}^{-1}. \tag{7}$$

With

$$\begin{aligned} \beta_0 &= 2I_3^{-1/2}\left(I_2 W_2 + I_3 W_3\right) \\ \beta_\varepsilon &= 2I_3^{-1/2} W_1 \\ \beta_{-\varepsilon} &= -2I_3^{1/2} W_2. \end{aligned} \tag{8}$$

The $W_{i=1,2,3}$ are the partial derivatives of W with respect to the invariants i.e $W_i = \partial W / \partial I_i$.
Backing to the relationships obtained in (5) with the hypothesis of incompressibility we mean $r = R$, we find:

$$\begin{aligned} I_1 &= 3 + \omega'^2; \\ I_2 &= 3 + \omega'^2; \\ I_3 &= 1. \end{aligned} \tag{9}$$

To write equilibrium equations, it is necessary to isolate a material domain and to apply to it the fundamental principle of dynamic. So then in the absence of volume forces, the equilibrium equation is given by:

$$div(\boldsymbol{\sigma}) = 0. \tag{10}$$

According to a study carried out in [6], the equilibrium equations are reduced to:

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rz}}{r} &= 0. \end{aligned} \tag{11}$$

By choosing as a condition to the limits on the inside radius R_i that in [6] and outside radius R_e that in [9] of the tube:

$$\begin{aligned} r(R_i) &= R_i; \qquad \omega(R_i) = 0 \\ \sigma_{rr}(R_e) &= 0; \qquad \sigma_{rz}(R_e) = \sigma_0. \end{aligned} \tag{12}$$

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where σ_0 is an initial constraint.
Starting from this previous reduction and with the necessary condition of shearing in the condition of incompressibilité with all the previous conditions, the solution of anti-planar shear ω is given in [6] by:

$$\omega = \frac{R_e \sigma_0}{W_1 + W_2} log(R) + C_0. \tag{13}$$

where C_0 is an integration constant.
Regarding the solution of the anti-planar shear ω, we consider its infinitesimal rate of an increasing radius of the order 10^{-3} or a great deformation, which means that I_1 and I_2 can be so close to the value 3 but will never be equal to 3 because this rate of change is always greater than zero.

3 Application

3.1 Model of Diouf-Zidi

Let's consider now the Diouf-Zidi model energy function defined in [10] by:

$$W = a_1 (I_1 - 3) + a_2 (I_2 - 3) + a_3 \left[\left(I_3^{1/p} - 1 \right) + (2-p)(I_3 - 1) \right] + a_4 \frac{2-p}{1+p} log (I_3) ; \tag{14}$$

where p is a positive real.
From this previous energy function, the condition of incompressibility yields us:

$$W = a_1 (I_1 - 3) + a_2 (I_2 - 3) . \tag{15}$$

Relation (15) allow us to obtain the solution of the anti planar shear and its derivative with no shear initially in the case of a Diouf-Zidi model of an incompressible material given by:

$$\begin{cases} \omega'(R) = \frac{R_e \sigma_0}{(a_1 + a_2)} \frac{1}{R}; \\ \omega(R) = \frac{R_e \sigma_0}{(a_1 + a_2)} log(R). \end{cases} \tag{16}$$

Here we see that the Diouf-Zidi model in incompressible allow us to obtain a logarithmic solution of the anti-planar shear in general, we mean in infinitesimal and great transformation.

3.2 Knowles-Sternberg Model

We consider here to be in the case of a transformation in anti-plane mode with the Knowles-Sternberg energy function by a power law [11]:

$$W = \frac{\mu}{2} \left(1 + \frac{b}{n} (I_1 - 3) \right)^n , \tag{17}$$

where μ and b are material parameters and n a strictly positive power.
Depending on the power, the local equations of movement are of a nature respectively elliptical, parabolic or elliptique-hyperbolic when the power n is respectively $> 1/2$, $= 1/2$ or $< 1/2$.
With the absence of the second invariant, the partial corresponding derivative becomes zero.
the computation of the partial derivative with respect to the first invariant gives:

$$W_1 = \frac{\mu b}{2} (I_1 - 3)^{n-1} , \tag{18}$$

The expression (18) allows us to have the general expression of the anti planar shear for the Knowles-Sternberg model.

$$\omega = \frac{2 R_e \sigma_0}{\mu b} (I_1 - 3)^{1-n} log(R) + C_0. \tag{19}$$

It should be noted that the shear will always be defined in the case of a purely longitudinal incompressible deformation because $I_1 - 3 \neq 0$ but so close to it in the case of an infinitesimal transformation.
So if we assume the case where the equations of motion are elliptical with $n = 3/2$, we finally find:

$$\omega = \frac{2 R_e \sigma_0}{\mu b} log(R) \omega'^{-1} + C_0. \tag{20}$$

The previous relationship with no initial shear gives us:

$$\omega\omega' = \frac{2R_e\sigma_0}{\mu b} log(R). \tag{21}$$

And then the solution of the shear and its derivative by the integration according to R with the condition that there is no shear initially becomes:

$$\begin{cases} \omega'(R) = \sqrt{\frac{R_e\sigma_0}{\mu b}} \left(R\left(log(R)-1\right)\right)^{-1/2} log\left(R\right); \\ \\ \omega(R) = \left(\frac{4R_e\sigma_0}{\mu b} R\left(log(R)-1\right)\right)^{1/2}. \end{cases} \tag{22}$$

We can see that the existential condition of the solution shows that only great transformations ($R > e = 2.7182818285$) are considered for this solution.

In the case of an infinitesimal transformation, we can pose $I_1 - 3 \approx \alpha \in \mathbb{R}$. So from the relationship (19) and with $n = 3/2$, we find these following expressions of the shears:

$$\begin{cases} \omega'(R) = 2\frac{R_e\sigma_0}{\mu\alpha b}\frac{1}{R}; \\ \\ \omega(R) = 2\frac{R_e\sigma_0}{\mu\alpha b} log(R). \end{cases} \tag{23}$$

So the power model of Knowles-Sternberg with $n = 3/2$ gives us a power solution of the shear with a logarithmic term in the case of great transformation but just a logarithmic solution of the anti-planar shear in the case of an infinitesimal transformation with certains conditions on the power in incompressible.

3.3 Delfino Model

We have defined in the case of a transformation in anti-plane mode a energy function by an exponential law [12]:

$$W = \frac{\beta_1}{2}\left[exp\left(\frac{\beta_2}{2}\left(I_1 - 3\right)\right) - 1\right], \tag{24}$$

where β_1 and β_2 are material parameters.
With also the absence of the second invariant, the partial corresponding derivative is zero.
The computation of the partial derivative with respect to the first invariant gives then:

$$W_1 = \frac{\beta_1\beta_2}{4} exp\left(\frac{\beta_2}{2}\left(I_1 - 3\right)\right), \tag{25}$$

The expression (22) allows us to have the general relationship of the anti planar shear for the Delfino Model.

$$\omega = \frac{4R_e\sigma_0}{\beta_1\beta_2} exp\left(-\frac{\beta_2}{2}\left(I_1 - 3\right)\right) log(R) + C_0. \tag{26}$$

The real exponential function can be characterized in a variety of equivalent ways. Most commonly, it is defined by the following power series:

$$exp\left(x\right) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{27}$$

To avoid an infinite behavior in the case of large shear variations, we restrict ourselves to the first three terms of this series, what gives us:

$$\omega = \frac{4R_e\sigma_0}{\beta_1\beta_2}\left(1 - \frac{\beta_2}{2}\omega'^2 + \frac{\beta_2^2}{8}\omega'^4 + O\left(-\frac{\beta_2}{2}\omega'^2\right)^3\right) log(R) + C_0. \tag{28}$$

In the scale of great transformation, we can see that this equation will not be solvable without any condition. That is why we consider a shear variation approximately equal to a real number ($\omega' \approxeq \eta \in \mathbb{R}$). So with this hypothesis and no initial shear, we obtain the following solution:

$$\begin{cases} \omega'(R) = \eta; \\ \omega(R) = 4\frac{R_e\sigma_0}{\beta_1\beta_2}\left(1 + \frac{\beta_2}{2}\eta^2\left(\frac{\beta_2}{4}\eta^2 - 1\right)\right) log(R). \end{cases} \quad (29)$$

In the infinitesimal transformation, hypothesis means that $\omega'^m \approx 0, \forall m \geq 2$. So from the equation (28) we obtain a simple expression of shear and its derivative of the Delfino model in this case with no shear anitially by:

$$\begin{cases} \omega'(R) = 4\frac{R_e\sigma_0}{\beta_1\beta_2}\frac{1}{R}; \\ \omega(R) = 4\frac{R_e\sigma_0}{\beta_1\beta_2} log(R). \end{cases} \quad (30)$$

As for the power model of Knowles-Sternberg, the Delfino model which has an exponential form gives also a logarithmic solution of the shear in the case of great deformaton by using the hypothesis of $\omega' = \eta \in \mathbb{R}$ and in the case of an infinitesimal transformation with certains conditions in incompressible.

Remark:

We find that the anti-plane shear solutions obtained for these three models in the case of an infinitesimal transformation are equivalent for certain conditions. In the Particular case where we have : $4(a_1 + a_2) = 2\mu\alpha b = \beta_1\beta_2$, we can see that these three solutions become identical.

In great transformation the solution found are different from a model to another and an hypothesis on the shear derivative of the exponential model is necessary for the recherche of its shear solution.

Our study shows that the form of a model has an influence on the shear only in great transformation because in infinitesimal transformation, shear solutions are all equivalent with our study conditions.

IV. Numerical simulation and interpretation

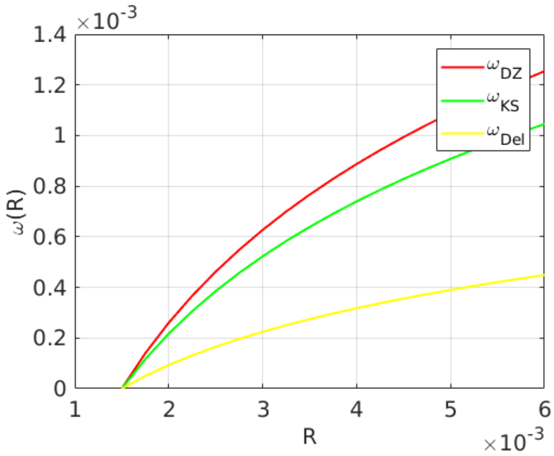
In this paragraph, we proceed to the simulation of expressions of anti-plane shear of the three models find previously. To do this work, we will consider the shear solution obtained for each model in great and infinitesimal transformationin order to see the behavior of the shearing in any cases.

We also consider the conditions that initially there is no shearing with an initial state of stress. These boundary conditions with the material parameters used are defined in the following table.

parameters	values
a_1	44.28 [KPa] [13,14]
a_2	21.79 [KPa] [14]
β_1	44.28 [KPa] [14]
β_2	16.7 [KPa] [14]
μ	44.28 [KPa] [14]
α	0.1
η	0.6
b	3.58 [KPa] [14]
σ_0	12.399 [KPa] [14]
R_e	0.482 [mm][14]

4.1 Infinitesimal Transformation Shears

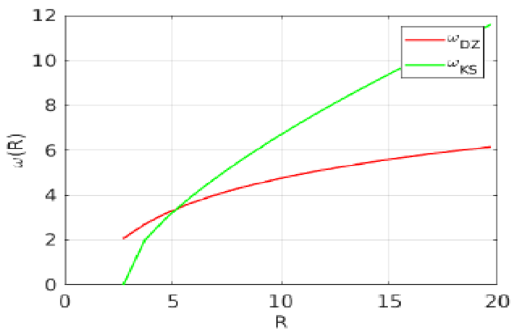
The simulation of the three infinitesimal solutions gives the following graphic:

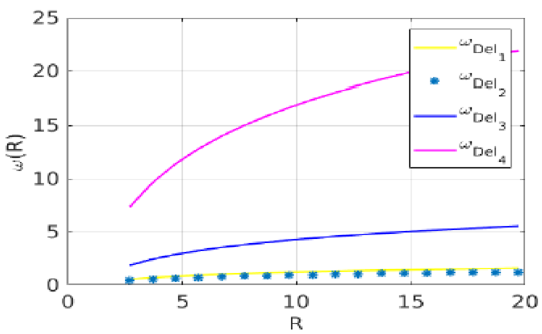


As the graphic shows, There is no shear initially, all three shears record the value zero. When the radius begins to grow, the shear rate found for the Diouf- Zidi model grows faster followed by the shear of the Knowles-Sternberg model with a difference which becomes biger and biger when the radius increases. this difference is in the order of 0.2 for radius of 6×10^{-3} between these both model.

The Delfino model follows that of Knowles-Sternberg with more difference be- tween them compared to that between Diouf-Zidi and Knowles-Sternberg. This difference is around 0.8 for a radius of 6 10^{-3}.
Our study shows that in infinitesimal transformation, we have the Diouf-Zidi model which records greater shear followed by the Knowles-Sterberg while the Defino model records the lowest shear. The difference observed between shears becomes more important with the increasing of the radius.

4.2 Great transformation Shears
The Diouf-Zidi and Knowles-Sternberg models are simulated in the same graphic because of its exact solutions. The Delfino model is simulate separately because of dependance on its derivative, four values of this last one are choosen. What gives us the two following graphics.





In great transformation, if we look at the comparison between the Diouf-Zidi model and that of Knowles-Sternberg, a meterial of Diouf-Zidi has the biger values of the shear than that of Knowles-

Sternberg when the radius of the tubular structure is smaller than 5 meters. But when the radius becomes bigerthan 5 meters, the reverse is observed in the graphic, we have the Shear ofKnowles-Sternberg model which is biger than that of Diouf-Zidi. We have tonote that a similitude is obtained between the both models at $R = 5$ meters. The value of the shear of Delfino model is proortional to the value of the shearderivative choosen by increasing with the radius. We can see that this last shearmodel can be smaller, between or biger according to the choosen value of itsderivative.
The simulations show that in great transformation, we mean $R > 5$ meters, the Knowles-Sternberg model is biger than the Diouf-Zidi model with those ofDelfino wich can be biger, between or smaller than these two models.

V. Conclusion

In this work of study of shear in a cylindrical tubular structure in great and infinitesimal transformation, we have used a fundamental solution of a shear obtained in our previous studies in the case of an isotropic material in incom- pressible. To do that, we used three energy functions of deformation which have polynomial, power and exponential form respectively.

In the first part reserved for calculations, this study allowed us to determine the integral solution of the shear for each models in the both case of transforma-tion. A solution of the logarithmic form for a certain choice on the parameters is always found in infinitesimal transformation. In great transformation, a log- arithmic solution is obtained for the Diouf-Zidi and Delfino models but a power solution with logarithmic term is found for the Knowles-Sternberg model.

The simulations shows that in infinitesimal transformation, we have the Diouf- Zidi model which records greater values followed by the Knowles-Sterberg whilethe Defino model records the lowest shear. In great transformation, we mean when the radius is greater than five meters, the Knowles-Sternberg model isbiger than the Diouf-Zidi model with those of Delfino wich can be biger, be- tween or smaller than these two others models.

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