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The Cauchy Stress Tensor – Calculation Parameters Via Matlab -A Review

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ABSTRACT
The presence of 2nd order tensors in problems in the broad area of Continuum Mechanics corresponds to a very common practice. In a general context, a solid body when subjected to external loads presents corresponding deformations, and the relationship between the acting stresses and the generated deformations depends on the constitutive relationship of the material considered. Metallic materials, in general, can be classified and grouped as isotropic materials. Within this entire context, the Cauchy stress tensor stands out, from which various calculations and analyzes can be developed. The main objective of this work is to present a general review of the calculation parameters that can be carried out using the tensor. Also noteworthy is the development of a simplified computational code, in Matlab language, which allows obtaining and analyzing the main calculation parameters to be considered based on the Cauchy stress tensor.

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I. INTRODUCTION

In certain disciplines in the broad area of Continuum Mechanics, it is necessary to know and apply the Cauchy stress tensor in the formulation of a wide variety of problems.It is a 2nd order symmetric tensor directly associated with the configuration or deformed condition (Eulerean formulation) of a given solid under analysis.The tensor is a direct consequence of the strain tensor and of the constitutive relationship corresponding to the material under analysis. For the general case of engineering problems, the materials considered correspond to materials classified as isotropic.

The present work therefore provides a detailed review of the obtaining and assembly of the Cauchy stress tensor based on the constitutive relationship of the material and the corresponding strain tensor. Subsequently, with the Cauchy stress tensor properly defined, it also becomes possible to obtain other tensors that are equally important in the solution or formulation of general problems of Elasticity and Continuum Mechanics, among others.Tensors covering the main stresses and their corresponding directions, maximum shear stresses, octahedral stresses and the tensors corresponding to the spherical and deviatoric portions can be adequately defined and thus evaluated based on the need or requirement of each specific problem.

Aiming to facilitate the calculations resulting from all the mathematical formulation involved, a simplified computational code in Matlab language was additionally developed, through which the end user is allowed to choose between different input options, the desired calculations for each specific problem. Consequently, and as described previously, all mathematical development was carried out using materials classified as isotropic, that is, metallic materials commonly used in engineering applications (carbon steels, aluminum alloys, etc.) that present their properties practically equivalent in all possible directions considered.

At the end of the paper, results arising from computer simulations associated with different problems that include the Cauchy stress tensor and its consequences are also presented.

II. MATERIAL AND METHODS

Using practical devices, known as deformation rosettes and/or simply strain-gages, which are commonly installed in parts of components subjected to external loads, it becomes possible to highlight and identify the value of the deformations suffered by the solid under analysis because of the external efforts. From this information and with the appropriate mathematical formulation, it is possible to obtain the strain tensor, which also corresponds to a second-order symmetric tensor that includes six independent strain components, three of which are normal strains (resulting from stretching or contractions) and three transverse deformations (resulting from of deformation in shear). Equation (1) reproduces the tensor with its corresponding components.

As, for example, $\varepsilon_{12} = \varepsilon_{21}$, the additional representation indicating the symmetry condition of the said tensor is then made.

This result is a consequence of practical applications carried out in terms of identifying and measuring deformations. In certain situations of conceptual problems, it is possible, of course, for this tensor to be provided directly as initial data for the problem.

$$\varepsilon_{ij} = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sim & & \varepsilon_{33} \end{vmatrix} \quad (1)$$

Constitutive Relations: Isotropic Materials

From the strain tensor and the constitutive relationship of the evaluated material, it becomes possible to identify the Cauchy stress tensor, the main proposal of the present work. The relationship includes two second-order tensors (stress tensor and strain tensor) and a fourth-order tensor designated as the tensor of the elastic constants of the material. For the specific case of isotropic materials, the tensor of the elastic constants contemplates only two parameters distinct of material, the rigidity modulus (G) and the Lamé constant (λ) and/or, alternatively, the longitudinal elasticity modulus (E) and the Poisson coefficient (μ). Equations (2) and (3) present these two alternatives.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{vmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2G+\lambda & \lambda & 0 & 0 & 0 \\ & & 2G+\lambda & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & G & 0 \\ sim & & & & & G \end{vmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{pmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ & & 1-\upsilon & 0 & 0 & 0 \\ & & & \frac{1-2\upsilon}{2} & 0 & 0 \\ & & & & \frac{1-2\upsilon}{2} & 0 \\ sim & & & & & \frac{1-2\upsilon}{2} \end{pmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (3)$$

As the information known about the material considered may differ from one problem to another, it was considered convenient to attribute to the proposed computational code the possibility of choosing between the known parameters of the material, thus providing the corresponding calculation of the other constants of the material. In addition to the four constants previously described, one can also consider the so-called Bulk's modulus (k). Table 1 reproduces these relationships.

Table 1: Relationship between constants - isotropic materials

E	ν	k	G	λ
$E = 3k(1-2\upsilon)$	$\upsilon = \frac{3k-E}{6k}$	$k = \frac{E}{3(1-2\upsilon)}$	$G = \frac{E}{2(1+\upsilon)}$	$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}$
$E = 2G(1+\upsilon)$	$\upsilon = \frac{E-2G}{2G}$	$k = \frac{GE}{3(3G-E)}$	$G = \frac{3kE}{9k-E}$	$\lambda = \frac{3k(3k-E)}{9k-E}$
$E = \frac{9k(k-\lambda)}{3k-\lambda}$	$\upsilon = \frac{3k-2G}{6k+2G}$	$k = \frac{2G(1+\upsilon)}{3(1-2\upsilon)}$	$G = \frac{3k(1-2\upsilon)}{2(1+\upsilon)}$	$\lambda = \frac{3k\upsilon}{1+\upsilon}$
$E = \frac{\lambda(1+\upsilon)(1-2\upsilon)}{\upsilon}$	$\upsilon = \frac{\lambda}{2(\lambda+G)}$	$k = \frac{3\lambda+2G}{3}$	$G = \frac{\lambda(1-2\upsilon)}{2\upsilon}$	$\lambda = \frac{2G\upsilon}{1-2\upsilon}$

Principal stress tensor

With the definition of the Cauchy stress tensor, it becomes possible, among other aspects, to define the stresses main and, subsequently, their corresponding main directions. This information becomes essential in the context of sizing and checking the component under analysis based on the design criteria conventionally used for this purpose, highlighting, among others, the Tresca and Mises criteria, for example.

The mathematical formulation corresponding to the calculation of principal stresses and directions contemplates a problem of eigenvalues (principal stresses) and eigenvectors (principal directions), falling into a third-degree equation (characteristic equation) whose roots correspond to the principal stresses (equation (4)).

The parameters I_1, I_2 e I_3correspond to the so-called main invariants of the stress tensor and are defined based on equation (5).

$$det(\sigma_{ij} - \sigma\delta_{ij}) = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \quad (4)$$

$$I_1 = \sigma_{ii} = tr[\sigma] = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (5)$$
$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) = \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{32} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{31} & \sigma_{33}\end{vmatrix}$$
$$I_3 = det[\sigma_{ij}]$$

Main directions

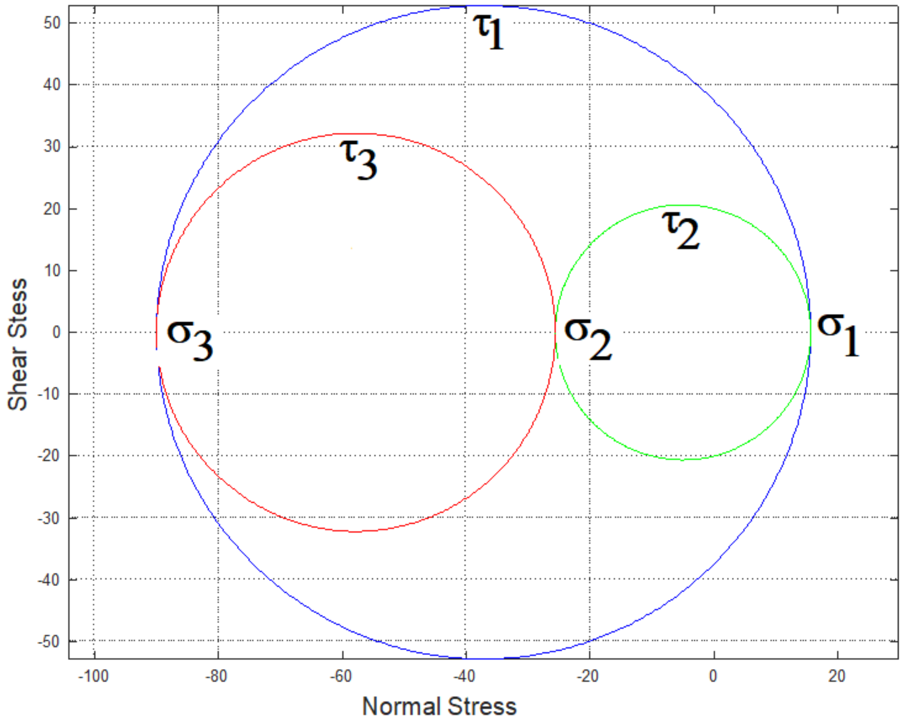
For each principal stress there will be its corresponding principal direction. The main directions (eigenvectors) include vectors, whose components n_1, n_2 and n_3 correspond to the directing cosines of each vector. The mathematical formulation corresponding to the calculation of each main direction contemplates a system of 3 equations with 3 unknowns, as identified by equation (6). It is also noteworthy that in certain circumstances, there will be the possibility of trivial solutions of the type, for example, $n_1 = n_2$, requiring the use of a complementary equation that encompasses a relationship between the directing cosines of this vector (equation 7).

$$\begin{vmatrix}\sigma_{11} - \sigma & \sigma_{12} & \sigma_{13}\\ & \sigma_{22} - \sigma & \sigma_{23}\\ sim & & \sigma_{33} - \sigma\end{vmatrix} \begin{vmatrix}n_1\\ n_2\\ n_3\end{vmatrix} = \begin{vmatrix}0\\ 0\\ 0\end{vmatrix} \quad (6)$$

$$n_1{}^2 + n_2{}^2 + n_3{}^2 = 1 \quad (7)$$

Maximum shear stresses

The Cauchy stress tensor arising from a given stress state can be graphically related to a 3D Mohr circle, which makes it possible to identify and visualize, in addition to the main stresses (σ_1, σ_2 and σ_3), the 3 maximum shear stresses, each associated with the radius of the corresponding Mohr circle. This information can be observed from figure 1 and mathematically by equation (8).



:

Figure 1: 3D Mohr's Circle

$$\tau_{max} = max = \begin{cases} \left|\frac{\sigma_1 - \sigma_2}{2}\right| \\ \left|\frac{\sigma_1 - \sigma_3}{2}\right| \\ \left|\frac{\sigma_2 - \sigma_3}{2}\right| \end{cases} \quad (8)$$

Octahedral stresses

Infinite imaginary planes, with their respective inclinations and directions, can be taken in relation to a given state of tensions. Among these different alternatives, a specific imaginary plane is considered which has the same inclination in relation to the three axes of a reference system whose axes contemplate the three corresponding main tensions (octahedral plane). This situation is schematized in figure 2.

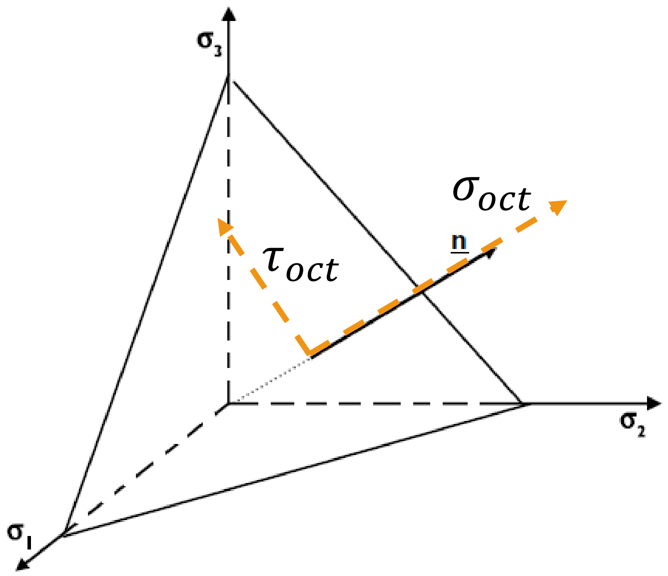


Figure 2: Octahedral planes and octahedral stress components

The angle formed in relation to the three reference axes corresponds to 54.4° (arc cos = 1/√3), while the normal and shear octahedral components are defined from equations (9) and (10), respectively. The importance of the octahedral shear component is highlighted, which is directly related to the Mises equivalent stress, a stress commonly considered in material plasticization studies.

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{\sigma_{ii}}{3} \quad (9)$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \quad (10)$$

Spherical and deviatoric stress tensors

The Cauchy stress tensor can also be divided into two other distinct tensors of order 2. These are the so-called spherical and deviatoric portions of the Cauchy stress tensor, or simply spherical tensor and deviatoric tensor. This division is important due to the physical conceptualization involved, that is, the spherical portion only includes deformations that cause variation in the volume of the solid under analysis, while the deviatoric portion includes deformations that cause variation in the shape and volume of said solid.

Again, this condition is important because only the stress deviatoric component is representative in the material's plasticization condition.Mathematically ($\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$):

$$\sigma_{sp\square} = \begin{vmatrix} \sigma_{ii}/3 & 0 & 0 \\ & \sigma_{ii}/3 & 0 \\ sim & & \sigma_{ii}/3 \end{vmatrix} \quad (11)$$

$$\sigma_{dev} = \begin{vmatrix} \sigma_{11} - \sigma_{ii}/3 & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} - \sigma_{ii}/3 & \sigma_{23} \\ sim & & \sigma_{33} - \sigma_{ii}/3 \end{vmatrix} \quad (12)$$

III. THE MATLAB CODE

The main purpose of this paper is to present a review of the main parameters and data that cover the construction and manipulation of the Cauchy stress tensor. It is easy to see, however, that the calculations involved can be relatively laborious and extensive, factors that can lead to relatively long times for the development of the calculations, as well as the possibility of obtaining inconsistencies in partial results and/or or endings.It was therefore proposed, and in a complementary way, the development of a computational code in Matlab (Matrix Laboratory) language that would allow speed in obtaining results, as well as guarantee the answers obtained for the most varied conditions and calculation proposals and simulations.

The developed code includes input windows, which allow the user to select: a) the input parameter (stress tensor or strain tensor – figure 3); and b) the parameters of the material contemplated (opting for any combination of 2 material parameters, among the available – figure 4).The code was also developed to perform operations in the international system of units (SI), as well as provide answers from a text file type, facilitating reading and interpretation of the finals results obtained.

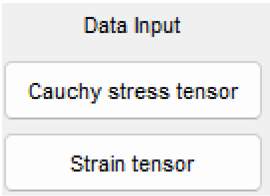


Figure 3: Matlab code initial (input) window

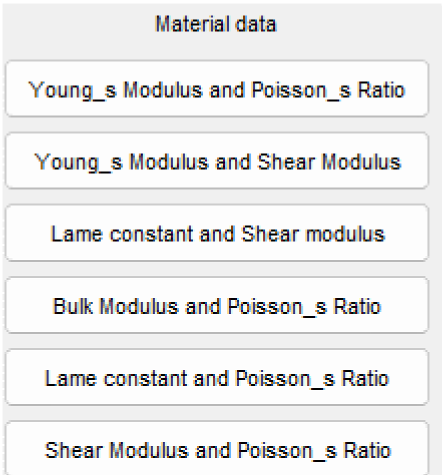


Figure 4: Material parameters input window

IV. CASE STUDIES

From the developed computational code, two different calculation examples are considered. The first example considers the Cauchy stress tensor as known (problem data), while the second example considers the strains tensor as known (problem data), in addition to parameters of the material considered (in this case, a carbon steel with E = 210 GPa and ν = 0.30). The set of responses is reproduced below.

Example 01:

Cauchy stress tensor (problem data)

```
----------------------
  150   100     0
  100   -50     0
    0     0   100
```

```
Principal stresses
----------------------
   -91.421         0         0
         0       100         0
         0         0    191.42

Main directions
----------------------
   0.38268
  -0.92388
         0

  0
  0
  1

  -0.92388
  -0.38268
         0

Maximum tangential stresses
----------------------
    45.711    141.42    95.711

Maximum tangential stress
----------------------
    141.42

Spherical stress tensor
----------------------
    66.667         0         0
         0    66.667         0
         0         0    66.667

Deviator stress tensor
----------------------
    83.333       100         0
       100   -116.67         0
         0         0    33.333

Normal octahedral stress
----------------------
    66.667

Shear octahedral stress
----------------------
    117.85

Resulting octahedral stress
------------------------
     135.4
```

<u>Example 02</u>:

```
enter the value of E:210e9
enter the value of Poisson:0.30
```

```
Modulus of Young
----------------------
210

Shear Modulus
----------------------
80.769

Poisson_s Ratio
----------------------
0.3

Lame Constant
----------------------
121.15

Bulk Modulus
----------------------
 175

Strain tensor(problem data)
----------------------
 0.0001    -5e-05     2e-05
 -5e-05     8e-05     3e-05
 2e-05      3e-05    -6e-05

Cauchy stress tensor
----------------------
30.692    -4.0385    1.6154
 -4.0385    27.462    2.4231
 1.6154    2.4231    4.8462

Principal stresses
----------------------
  4.4284        0        0
0    25.145        0
0        0    33.426

Main directions
----------------------
-0.079025
-0.11799
0.98987

0.55524
0.81948
0.14201

-0.82793
0.56084
0.00075346

Maximum tangential stresses
----------------------
4.1407    14.499    10.358

Maximum tangential stress
----------------------
14.499
```

Spherical stress tensor
----
21 0 0
0 21 0
0 0 21

Deviator stress tensor

9.6923 -4.0385 1.6154
-4.0385 6.4615 2.4231
1.6154 2.4231 -16.154

Normal octahedral stress

21

Shear octahedral stress

12.196

Resulting octahedral stress

24.285

V. DISCUSSION AND CONCLUSION

Problems in general, involving 2nd order tensors, require, as a rule, laborious mathematical developments that are often prone to errors because of the various steps to be considered. Within this context, when considering the Cauchy stress tensor, there are different calculation possibilities to be considered, depending on the focus of the problem under analysis. Disciplines of Elasticity, Solid Mechanics, Continuum Mechanics and even Plasticity often requires data manipulation and calculations with the Cauchy stress tensor. It is understood, therefore, that the proposed review, in addition to facilitating and contributing to the understanding and consolidation of these concepts, favors the calculations to be carried out using the jointly developed and proposed computational code.

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