International Journal Of Engineering Research And Development
e- ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 20, Issue 8 (August, 2024), PP. 477-487

An Algorithm to Choose the Best Finite Bundle of Products: HB Algorithm

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Choosing a particular good or service always depends on the utility that we can expect from it. In different activities, a moment arrives when we need to choose one specific function, object, material, or generally, an entity, comparing to other similar entities based on the maximum efficiency that can be obtained with the least expenses of inputs. In economics, there are two famous problems on this same situation, namely, the Expenditure Minimization Problem (EmP) and the Utility Maximization Problem (UMP). However, there is a solution curve to both of these problems and when the curves of the individual functions attain monotonicity, we obtain a solution of UMP and EmP. This work focuses on establishing a generalized function for computing maximized utility of a function with minimum expenses. Further it gives a suitable approach that can be used to find the best finite bundle of products for a consumer.

Date of Submission: 23-08-2024 Date of Acceptance: 03-09-2024

I. INTRODUCTION

In microeconomics and consumer theory, the expenditure minimization function has been a huge addition to the domain. This helps in a decision-making of the user while purchasing a service. If we see in mathematical sense, there are multiple algorithms and approaches to a particular problem. Suppose, we know the number of inputs required for each algorithm to solve this particular problem is known. Now, based on our preferences and a given constraint number of inputs, we must choose the most efficient algorithm out of the set. However, choosing a particular entity, be it a function or a service depends on the budget or the total estimated inputs we wish to invest to get the output and also depends on preference along with its utility. However, keeping the budget fixed, there are two different conditions. One is, what is the maximum utility of a particular entity that one can expect and the other is, provided that we have a fixed expected utility, what is the minimum input we need to provide. However, in order to fulfil and have an optimal solution where we actually have a fixed budget and we pay a minimal input to get the maximum possible utility, we get a supply to the "demand" of such a function or service. Talking in context, the functions of EmP and UMP arrive at a monotonicity when we get such a solution. Based on the constraints of demands (fixed utility, fixed expenditure and others), there are different demand functions namely – the Hicksian demand, the Marshallian demand and the Slutsky demand. Catering through all these demands, the purpose of this paper is establishing an algorithm and set of expressions to find the maximum utility and the minimum price one needs to pay for achieving that utility. However, further using Statistical techniques, this helps in figuring out the best finite set of product bundles or entity bundle that can be obtained and have equivalent effect of utility and lesser expenses involved. This selection helps in multiple domains apart from its core fields like Economics, Data Science and Applied Statistics and also helps in choosing the right bundle for optimization problems in Engineering where several parts, materials, algorithms or fluids are needed to be chosen from a huge set of available resources.

II. DEMAND

In general, demand can be termed as the willingness to avail a good or service or approaching a particular function. In other words, when a particular function is approached for solving a problem, a "demand" is created for that function. This demand is based on utility, expenditure, preferences and budget. An equilibrium of utility, expenditure and preference sets a significant demand for the entity. To cater the quantity of service needed to be availed, there are different demand functions that are being used.

Demands can be categorized based on different scales of requirement, quantity and different constraints. The different demands involve Market demand (cumulative demand of a product in a market), Individual demand (demand by a single consumer), Cross demand (demand of good when the price of another good is changed), Price demand (demand that varies due to price of the good), Direct demand (demand for goods and services that are directly consumed by the individuals to satisfy needs and wants), Derived demand (demand of a particular good or service due to increased demand of another good or service), Income demand (demand based on income level of consumers), Composite demand (demand of a good that has multiple uses), Latent demand (demand for a good or service whose desire has not currently been met in the market due to lack of availability or awareness), Elastic demand (demand that varies highly with change in price), Inelastic demand (demand that varies less with change in price and usually caters needs of the individual) and Aggregate demand (total demand of goods and services in a given time period at a price level in an economy).

Wu (2023) has explained how the type of good influences income that affects budget of a consumer and in turn manipulates demands. In fact, not only economically, the impacts of demand are driven through psychological and artificially nudged economic behavior towards the market. Rachlin et al. (1976) showcases two experiments with rats to show how consumption of commodities changed when changes were introduced to the budget. Needs and wants, in other words, essential and non-essential goods consumption were affected and substitution patterns were observed by reinforcement in the budget.

This leads us to thinking on the point that how can demands be compensated or met. Based on the compensation of demands, different demand functions have been concluded, namely, the Hicksian demand function, the Marshallian demand function and the Slutsky demand function. The Demand functions are derived from Indirect Utility functions and Expenditure functions. These functions help us to analyze the scenario that are required to fulfil the demands of the consumers for a particular good or service possessing some utility.

III. UTILITY AND ITS FUNCTIONS

Utility in general defines the value of goods and services. Utility can be equated to the state of satisfaction, well-being or happiness but does not possess an exact definition [1]. Kapteyn (1985) has plotted the measurability of utility as an entity and mentions it as an ordinal quantity where a higher value representing utility in the plot corresponds a higher utility. The biggest point in computing utility as a statistical value is to use it as a metric to satisfy as a demand. The factors affecting the value of utility of a product involve income, budget, preference of a product by a consumer, price of the product and marginal utility of each product. When we have multiple such products in the market, we compute the utility using the mentioned factors. There are specifically constructed functions take into consideration these factors and output is the utility of the targeted product. These functions are called utility functions and are mainly of three types: Cobb – Douglas, CES and quasi – linear [5].

3.1 **Cobb – Douglas Function**

The Cobb – Douglas function deals with products and their preferences on a particular utility scale that determines the utility that can be derived by using that particular combination of products. Considering the traditional form of the utility function, we get the following format:

$$U(x_1, x_2) = A x_1^{\alpha} x_2^{\beta} \qquad (3.1)$$

where $U(x_1,x_2)$ (also denoted as U^*) is the derived utility, A is a non-negative constant representing scalability factor, x_1 and x_2 representing the quantity of goods used (can be price or physical quantity) and the constants α and β represent the relative preferences of each of the commodities. It is to be noted that the sum of relative preferences is always 1 and hence $\alpha + \beta = 1$. However, we can write a generalized function for it where we consider it for n products. Defining the consumption quantity of the products as $x_1, x_2, ..., x_n$, their corresponding relative preferences being $\alpha_1, \alpha_2, \ldots, \alpha_n$, we derive at equation (3.2a) that follows the conservation rule stated in equation (3.2b)

$$U(x_1, x_2, .., x_n) = A \prod_{i=1}^{n} x_i^{\alpha_i} \qquad (3.2a)$$

Where,

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad (3.2b)$$

The utility when two different products are in use can be plotted as a function of two independent random variables. If we consider the scalability constant as 1 and the relative preferences for two products being 0.7 and 0.3 (values of α and β respectively), we get the following behavior as plotted in Figure 1.

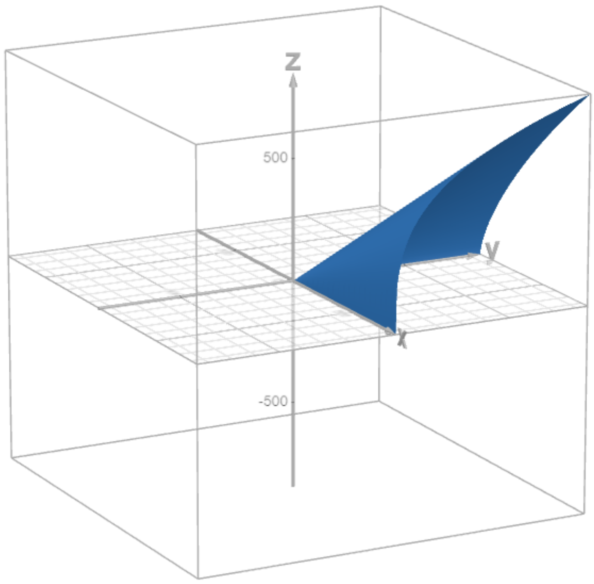

Figure1: Graphical Representation of Cobb-Douglas Utility function

Figure 1 plots the utility U^* along z-axis, product with preference 0.7 along x-axis and product with preference 0.3 along y-axis. Clearly observing the output as plotted along the z coordinates, the function is concluded to be of monotonically increasing nature.

3.2 CES Utility Function

Constant Elasticity of Substitution (CES) is the property which states that proportional changes in relative prices to relative quantities is constant. The expression for this function for a set of n commodities from x_1 to x_n with relative preferences α_1 to α_n such that $\sum_{i=1}^{n} \alpha_i = 1$. Along with these, there is an elasticity parameter ρ which is a numerical parameter that determines how easily can we substitute a product with another. It is derived from the elasticity of substitution which denotes the willingness of consumers to substitute one good with another. Elasticity of substitution is given by

$$\sigma = \frac{1}{1-\rho} \qquad (3.3)$$

CES utility function can be written as a linear combination in the following way

$$U^* = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} \qquad (3.4)$$

This can be written as a more complex linear transformation where we also add the scalability factor. In that case we can consider the individual preferences to be β_1 to β_n. Keeping the scalability factor as A, CES can be written in another way as

$$U^* = A\left(\sum_{i=1}^{n} \beta_i x_i^{\rho}\right)^{\frac{1}{\rho}} \qquad (3.5)$$

The nature of this function can be seen in Figure 2. Similar to Cobb-Douglas Utility, the CES Utility function is also monotone increasing function. For a simpler visualization, a three-dimensional graph is created consisting of two products x_1 and x_2 with preferences 0.7 and 0.3 respectively. The elasticity parameter is set to m. For simple computations, scalability factor A is set to 1. The different subparts of Figure 2 show the condition at a particular value of m.

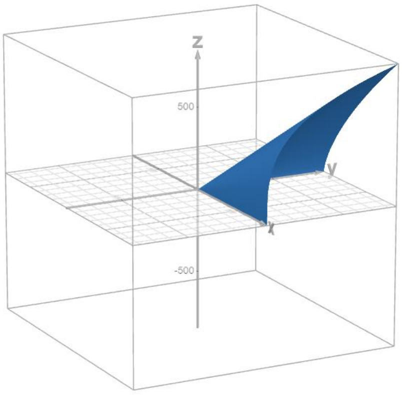

Figure 5a: CES Utility Function at m=-1.1

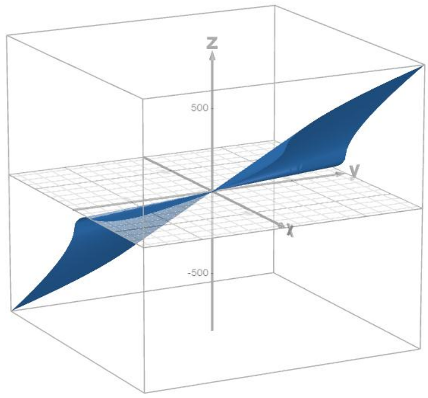

Figure 4b: CES Utility Function at m=0.6

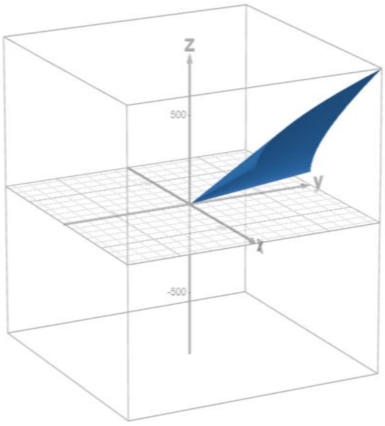

Figure 3c: CES Utility Function at m=0.5

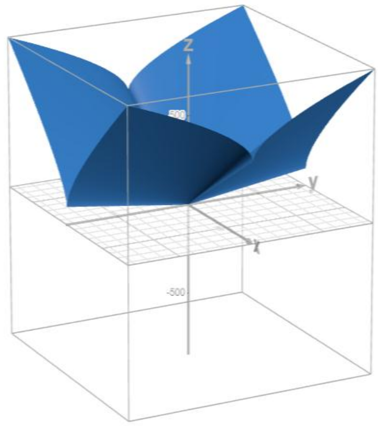

Figure 2d: CES Utility Function at m=0.8

It is observed that the graphs are repeating periodically with a constant multiplicative factor. For instance, if we take a non-zero and non-negative real number n, then the nature of graph is similar to figure 2b at m=6n. Similarly figure 2d is observable at m=8n. However, this is to be noted that m cannot be a negative value and Figure 2a is a theoretical scenario.

3.3 Quasi-Linear Utility Function

As the name suggests, this utility function is combination of a pre-existing function and variables. The pre-existing function may not be linear but the resulting function makes it a linear structure hence, quasi-linear. Considering a set of n products from x_1 to x_n, a pre-existing function can be written till x_{n-1} and the quasi-linear expression can be written as

$$U^* = f(x_1, x_2, \dots, x_{n-1}) + cx_n \qquad (3.6)$$

where c is a constant. However, instead of one single function it could be combination of multiple functions some of whose forms are represented in equations 3.7 and 3.8. Equation 3.7 shows that if all variables till x_{n-1} contains their own function f(x), then U^* is represented in the following way

$$U^* = \sum_{i=1}^{n-1} f(x_i) + cx_n \qquad (3.7)$$

Now, suppose there are different functions with different combination of variables which can be used to form a quasi-linear utility function expression as shown in equation 3.8.

$$U^* = f(x_1, x_2) + f(x_3, x_7, x_{11}) + f(x_4, x_9) + \cdots + cx_n \qquad (3.8)$$

However all composite forms of this expression can be simplified and written in a form as shown in equation 3.6.

All the forms of utility functions that have been seen so far can be manipulated and used to derive other utility functions, compute other economic terms like expenditure and demand and also can be interchanged among each other. For instance, if we want to compute Marshallian demands for two products x_1 and x_2 with equal preferences α, the utility maximization problem with CES utility function can be stated as

$$\begin{cases} \max u(x_1, x_2) = x_1^\alpha + x_2^\alpha \\ subject\ to\ p_1x_1 + p_2x_2 = y \end{cases}$$

where p_1 and p_2 are prices of the corresponding products. Now forming the Lagrangian function L for the same to compute the first order condition, we get

$$L = x_1^\alpha + x_2^\alpha - \phi(p_1x_1 + p_2x_2 - y)$$

with ϕ as the Lagrangian multiplier, the conditions are

$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1} - \phi p_1 = 0 \qquad (3.9)$$

$$\frac{\partial L}{\partial x_2} = \alpha x_2^{\alpha-1} - \phi p_2 = 0 \qquad (3.10)$$

hence computing the Marginal Rate of Substitution (MRS), we get $\frac{p_1}{p_2}$ by simplifying and dividing equation 3.9 by 3.10. The resulting MRS is hence $\left(\frac{x_1}{x_2}\right)^{\alpha-1}$. On the other hand, maximization on Cobb – Douglas will have the following expression for utility maximization

$$u(x_1, x_2) = x_1^\alpha x_2^\alpha$$

Now computing the indifference curve for the Cobb – Douglas form, we get to see that the indifference curves satisfy the MRS = $\frac{x_1}{x_2}$ [5] which imply that indifference curve of Cobb – Douglas depends on proportional values of the products and not the absolute value of the products. Similarly when MRS of CES has α value 0 then MRS is $\left(\frac{x_1}{x_2}\right)^{-1}$ or $\left(\frac{x_2}{x_1}\right)$. Hence the indifference curve is similar to that of Cobb – Douglas indifference curve and hence at 0 preference, CES becomes Cobb – Douglas. Further a brief into indirect utility and demand functions will lay the fundamentals to the maximization and minimization problems.

IV. THE INDIRECT UTILITY FUNCTION – OUTPUT TO UMP

The indirect utility function lays the foundation to a well-known problem of consumer theory known as the Utility Maximization Problem. First studied and discovered by Antonelli (1886), it gives the maximum utility that a consumer can achieve by using a particular product for a given price of the product and a given income of the consumer. In fact, mathematically, it is expressed as a function of income and price of goods. Sakai (1977) shows a duality of direct and indirect utility functions through axioms [7]. Briefly, the direct utility function is a computational output that expresses utility as a quantitative measurement. In other words, it is a function that provides utility of a product based on the quantity consumed. However, the indirect utility function is dependent on direct utility to some extent and can be expressed mathematically as shown in equation 4.1. If we consider a set of n products with prices from p_1 to p_n, consumed quantities x_1 to x_n and income I, then the indirect utility function $V(p_1,p_2, \dots , p_n, I)$ as a function of direct utility function $U(x_1, x_2, \dots , x_n)$, price, quantity and income will be expressed as

$$\begin{cases} V(p_1, p_2, \dots, p_n, I) = max_{x_1,x_2,\dots,x_n} U(x_1, x_2, \dots, x_n) \\ \qquad subject\ to\ \sum_{i=1}^{n} p_i x_i \le I \end{cases} \qquad (4.1)$$

Some of its remarkable properties lead to utility maximization. One of these involve the nature of non-decreasing utility nature with increasing income. As per this, the utility remains constant or increases as the income increases. However, if the price increases, the utility remains constant or decreases and hence providing a non – increasing nature of prices. The term of 'indirectness' is directly related to the behavior of consumers in the aspect of 'utility' where they think that direct utility is the associated function for choosing the bundle of products where choice is based on the quantity of products consumed. However, indirect utility shows the impact of price and income that is involved in choosing the perfect bundle. The utility functions that have been studied in section III can be used as a direct utility function which can be further utilized to derive the indirect utility. This justifies that direct utility can be Cobb – Douglas, CES or quasi – linear. However, observing the nature of the indirect utility function, it is quasi – concave in income, hence showing concave nature when derived from income. Hence, with an increasing indirect utility, marginal utility decreases. Observing the indirect utility for different forms of utility functions will help us to observe and study how different forms of utility functions affect the indirect utility. For Cobb – Douglas utility involving two variables, preferences α and β, if income is I and prices are p_1 and p_2 respectively for consumed quantities x_1 and x_2, utility can be computed using the expression derived in equation 4.2. For deriving the expression, the value of A is set to 1 and the variables are substituted in equation 3.1 which is used for the base function. Further, Lagrange Multipliers are used for each product to derive the optimal utility of consumption for each product (denoted as x_1^* and x_2^*) and finally these values are substituted from the expressions of $\frac{\partial L}{\partial x_1}$ and $\frac{\partial L}{\partial x_2}$ to get x_1^* and x_2^* and finally the indirect utility expression is achieved. Considering CES, if consider the products x_1 and x_2 with importances 'a' and 'b' and prices p_1 and p_2 respectively, then the indirect utility for elasticity ρ is derived in equation 4.3. For constant elasticity, we use Lagrangian partial derivatives to find the maximized utility expression and hence derive the expression for optimal consumption.

This is given by $x_1 = \left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} \times \frac{I}{p_1\left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} + p_2}$ and $x_2 = \frac{I}{p_1\left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} + p_2}$. Using this in the utility expression where n is set to 2 and all assumptions are mentioned, then we get the expression for indirect utility when the utility is CES.

$$V(p_1, p_2, I) = \left(\frac{\alpha^{\alpha}\beta^{\beta}I^{\alpha+\beta}}{p_1^{\alpha}p_2^{\beta}}\right) \qquad (4.2)$$

$$V(p_1, p_2, I) = \left(\left(a\left(\left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} \times \frac{I}{p_1\left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} + p_2}\right)^{\rho}\right) + \left(b\left(\frac{I}{p_1\left(\frac{p_2 a}{p_1 b}\right)^{\frac{1}{\rho-1}} + p_2}\right)^{\rho}\right)\right)^{\frac{1}{\rho}} \qquad (4.3)$$

Further the expenditure function will be explained and using indirect utility and expenditure function to derive the demand functions and study the demand functions. These will lead to the two highlighted problems whose solution can be found using the algorithm proposed further.

V. EXPENDITURE FUNCTION – THE OUTPUT OF EmP

Put forward in the Consumer Theory by Deaton (1986), the expenditure function is the centric model for computing utility and preferences and plays an important role in Demand Theory as it combines with the indirect utility function to create the demand functions and hence obtain the demand curves. In general, it helps in computing the minimum amount that a person needs to pay to achieve a particular utility level for a product. In simple terms consider a utility function U(x) which gives the utility limit U and an n – dimensional price vector $\vec{p} = (p_1, p_2, \dots, p_n)$ for a quantitative vector $\vec{x} = (x_1, x_2, \dots, x_n)$ implying quantity of goods, then the expenditure function $e(\vec{p}, U)$ is given by

$$e(\vec{p}, U) = \min_{x}\{\vec{p} \cdot \vec{x} : U(\vec{x}) \geq U\}$$

However, we can rewrite the above equation by set of tuples such that e_p is a binary relation between price and quantity which can be topologically represented on a convex cone where $\vec{p}$ and $U(\vec{x})$ are two vectors representing price and utility limit (quantitative for quantity x) on the topologically convex vector space ξ that contains the convex cone X. Then e_p is a reflexive and transitive binary relation termed as preorder and is represented as

$$e_P(\vec{p}, U) = \bigwedge_{\vec{x} \in X}\{\vec{p} \cdot \vec{x} | U(\vec{x}) \geq U\}$$

here U is the desired utility level. There are several works, first led by Divisia (1928) that shows duality of expenditure function and indirect utility function. Glancing into the properties of the expenditure function, it possesses a non – decreasing nature with prices and hence increases as price of product(s) increases. However, it also shows concavity with prices as the prior goal of consumers is to minimize the expenditure to get their desired utility. At degree 1 of expenditure function, homogeneity is observed with prices. Checking the expenditure function for two products x_1 and x_2 with relative preferences α and β, prices p_1 and p_2, having the utility level u, then, for Cobb – Douglas, the utility function is equation 3.1, and the expenditure function becomes

$$e(p_1, p_2, u) = u \cdot p_1^{\alpha} \cdot p_2^{\beta}$$

Similarly, for CES, when elasticity is ρ, the utility function is given by equation 3.4, if n is set to 2 and all the variables have same definition as in equation 3.4 with prices p_1, p_2 and utility level being u then the expenditure function is given by

$$e(p_1, p_2, u) = u\left(\frac{p_1^{\frac{\rho}{\rho-1}}}{\alpha_1^{\frac{1}{1-\rho}}} + \frac{p_2^{\frac{\rho}{\rho-1}}}{\alpha_2^{\frac{1}{1-\rho}}}\right)^{\frac{1-\rho}{\rho}}$$

This function can be used in optimizing costs of consumption based on desired utility levels and hence can be used when dualled with indirect utility function to find the minimum cost required to get the maximum utility from any entity. Applications of expenditure minimization vary from using it in conventional ways of cost analysis for different products in the financial market to finding the minimum weighted path used to traverse a tree in graph theory.

In Sections IV and V, we have studied the backbone functions for the Utility Maximization Problem and the Expenditure Minimization Problem respectively. Briefly looking into the statements of UMP and EmP, the importance of demand curves and their types will be studied, that will further lead to the creation of the algorithm.

VI. UMP & EmP: A BRIEF OVERVIEW

Considering the most important two problem statements from the 20th century which created the major segment of Consumer Theory, supported the Demand Theory and finally the whole branch of Microeconomics, the Utility Maximization Problem and the Expenditure Minimization Problem will be briefly discussed here.

6.1 Utility Maximization Problem: This particular problem, simply provides the maximum utility, that a consumer can afford, provided that they have a fixed income or a budget. This willingness to consume the particular utility is the phenomenon which we call as a demand. Now, there are a few parameters which complete the checks of a demand that can be fulfilled using UMP. The first check is following the trajectory set by a French economist Léon Walras (1874) who stated the Walras Law [17] where one checks three states of preferences of a consumer to judge the existence of the utility to fulfil the demand. So, the preferences of the consumer must by a complete preference firstly. In this, either the products that are being compared in the bundle are indifferent (i.e., does not contain any distinct difference) or the consumer must have a distinct preference of one product over the others. Hence for a set of n products $\{A_1, A_2, \dots, A_n\}$ there must be either a product A_i which has the highest preference or all the products must be indifferent. Secondly, a monotonic preference must be existing. Considering the set of n products, if we consider a product A_i such that, a minor increment in quantity of A_i is made, then the preference is monotonic if the consumer prefers A_i more than others. Strict preference to higher quantity with same utility level must be existing to conserve the preference. Equation 6.1a and 6.1b are the conditions for monotonic preference. Further, once monotonic preference is established, a transitive preference is also needed for ensuring consistency in preferences of the consumer.

$$\{A_1, A_2, \dots, A_i + \delta, \dots, A_n\} \succcurlyeq \{A_1, A_2, \dots, A_i, \dots, A_n\} \quad (6.1a)$$

$$\{A_1 + \delta, A_2 + \delta, \dots, A_i + \delta, \dots, A_n + \delta\} \succ \{A_1, A_2, \dots, A_i, \dots, A_n\} \quad (6.1b)$$

Following all these three states of preferences confirm validation of Walras law. Now, if the Walras law is validated, then the optimal demand lies along the budget line, which represents the budget constraint, one of the four checks to see whether UMP is applicable or not. The budget constraint is expressed in equation 6.2. Further, when this optimal solution is existing, the Lagrangian functions for the utilities are partially differentiated to find the Marginal utility from tangential conditionality and a third check is done which confirms that the ratio of Marginal utility of a product to the price of the product is constant within the vector space containing the marginal utilities and the prices of the products (equation 6.3). Finally, the budget constraint is set as per the income of the consumer and completes the fourth check for validating the existence of UMP. The prices are defined as $\{p_1, p_2, \dots, p_n\}$ for corresponding products, income is taken as I and quantity of consumption of each product is given by $\{x_1, x_2, \dots, x_n\}$.

$$\sum_{i=1}^{n} p_i x_i \le I \quad (6.2)$$

As per Walras law, when the law is satisfied, then 6.2 is modified to

$$\sum_{i=1}^{n} p_i x_i = I$$

Further, the tangential conditionality gives a check to the phenomenon transited from defense to economics known as 'Bang for your buck'[19] and is given by

$$\frac{MU_{A_1}}{p_{A_1}} = \frac{MU_{A_2}}{p_{A_2}} = \dots = \frac{MU_{A_n}}{p_{A_n}} \quad (6.3)$$

This gives the Utility Maximization Problem, which considers a utility function $U(x_1, x_2, \dots, x_n)$ and the budget constraint as explained in equation 6.2 and combines them to give

$$\begin{cases} \max U(x_1, x_2, \dots, x_n) \\ subject\ to \displaystyle\sum_{i=1}^{n} p_i x_i \le I \end{cases}$$

The output provides a bundle $\{x_1^*, x_2^*, \dots, x_n^*\}$ which contains those products which combine to give maximum utility to the consumer.

6.2 Expenditure Minimization Problem: The expenditure minimization problem or the EmP, which is the dual of UMP, provides the expenses one needs to make to achieve the desired level of utility. First laid into discussion by Paul Samuelson (1947), this analyzes the desired utility that a consumer wants to achieve, provided that they have a particular utility level and needs to spend the minimum possible amount for the same. However, addressing the numerical amount that the consumer needs to pay to avail these utilities is given by the expenditure function of section V and the product bundle that could be suggestively listed to spend the minimum is given by the Hicksian demand which is discussed in topic 7.1. The statement for EmP can be mathematically be written as

$$\begin{cases} \min_{x_1,x_2,\dots,x_n} \sum_{i=1}^{n} p_i x_i \\ subject\ to\ U(x_1, x_2, \dots, x_n) \geq U_0 \end{cases}$$

where $\{x_1, x_2, \dots, x_n\}$ is the set containing quantity consumed for each product, $U(x_1, x_2, \dots, x_n)$ is the utility function for the n products and U_0 is the desired or minimum expected utility limit. The demand that is being fulfilled here has a limiting utility that needs to be checked for getting the most desirable output. Even when there is a change in prices, the utility limit is checked such that it remains the same or is higher than the lower utility limit. Hence, we can say that all demands are compensated and the compensation variation is checked upon. Most of the outputs of the EmP lies with the nature of demand and hence a wider analysis into the demand curves will help in understanding the nature of outputs from these two problems and hence understanding the functionality of the HB algorithm which will be further declared.

VII. DEMAND FUNCTIONS

Using the indirect utility function and the expenditure function from the sections IV and V, the demand functions are obtained. Plotting these functions give the demand curves which helps us in justifying the nature of the demand, optimal utilities and solutions to different points. These play a very vital role in behavioral economics where variations in the demand curves help in determining the effect of utility, quantity or other specific variations on the consumption pattern of products. These variations are often termed as elasticities which vary due to income, price of product or even cross – price which means that due to change in price of one product, consumption quantity of another product may get affected. There are broadly three types of demand functions which one can find in economics, namely – Hicksian demand, Marshallian demand and Slutsky demand. Out of these, the Hicksian demand and the Marshallian demand are the most studied functions as they form the backbone for this domain. Insights into these functions will help us understand the solutions of UMP and EmP and further the form of solution that is expected from the HB algorithm.

7.1 Hicksian Demand: The Hicksian demand or the compensated demand is the phase where the consumer gets to know which bundle of products need to be chosen, provided that the target minimum utility is constant and the price paid for the same is the minimum. Named after John Hicks, this function is the solution for the Expenditure Minimization Problem. The main focus of this demand function is to address how the expenses can be minimized without compromising the utilities. Mathematically stating the Hicksian demand can be stated as

$$h(\vec{p}, U) = arg \min_{\vec{x} \in X} \{\vec{p} \cdot \vec{x} | U(\vec{x}) \geq U\}$$

where $h(\vec{p}, U)$ is the demand function and $\vec{p}$ and $\vec{x}$ are the price and quantity vectors contained in a topological vector space X where $X = \mathbb{R}^n_+$. Similarly, U is the minimum limit of utility contained in the vector space X and $h: \mathbb{R}^n_+ \to \mathbb{R}$. $U(\vec{x})$ is the utility function which keeps a check on the utility limit. The function can be derived from expenditure function by using Lagrangian multiplier. Provided that the vectors and variables remain the same, using EmP, if a Lagrangian multiplier is denoted by φ, the Lagrangian function $\mathcal{L}$ is given by

$$\mathcal{L}(x_1, x_2, \dots, x_n, \lambda) = \sum_{i=1}^{n} p_i x_i + \varphi(U - U(x_1, x_2, \dots, x_n))$$

checking in the first-order conditions for the function with respect to i-th product, we get

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \varphi \frac{\partial U}{\partial x_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = U - U(x_1, x_2, \dots, x_n) = 0$$

Hence, we get price of the i-th product to be

$$p_i = \varphi \frac{\partial U}{\partial x_i}$$

which gives φ as $\frac{p_i}{\frac{\partial U}{\partial x_i}}$ which is constant for all cases. Note that $\frac{\partial U}{\partial x_i}$ is the marginal utility for the i-th product. To get the Hicksian demand $h_i(\vec{p}, U)$ for the i-th product, it applies to the utility constraint such that

$$U(h_1, h_2, \dots, h_n) = U$$

This, in turn, gives the set of outputs for all h_i for $1 \leq i \leq n$. This is one way of deriving the Hicksian demand by using utility constraint. Another way is using Shepherd's Lemma which uses the expenditure function $e(\vec{p}, U)$ and states that the positive gradient of the expenditure function with respect to price of i-th product is h_i. It is given by

$$h(\vec{p}, U) = \nabla_{p_i} e(\vec{p}, U)$$

this is the compensated demand or Hicksian demand and further we will see how is it different from the ordinary demand or the Marshallian demand by studying the Marshallian demand. However, correlating to the algorithm that has been proposed, it uses Hicksian demand as the solution.

7.2 Marshallian demand: Ordinary demand or Marshallian demand, which is named after economist Alfred Marshal, which provides information to the consumers on how to choose a product bundle such that the utility is maximized. Unlike Hicksian demand, this is uncompensated demand and its main focus is to maximize the utility of the consumer. In fact, many times this is considered as the solution for UMP. Mathematically, it can be expressed as

$$x_i^m(\vec{p}, I) = arg \max_{\vec{x}\in X}\{U(\vec{x}) | \vec{p}\cdot\vec{x} \leq I\}$$

where I is the income of the consumer, $\vec{p}$ and $\vec{x}$ are two vectors for price and quantity of consumption respectively with utility function U($\vec{x}$) all belonging to topological vector space X where X = $\mathbb{R}_+^n$ and Marshallian $x_i^m\colon \mathbb{R}_+^n \to \mathbb{R}$. Looking into the derivation of this function, Lagrangian Multiplier form can be used to create a Lagrange function utilizing the budget constraint as per equation 6.2, which gives the function L_m.

$$L_m = U(x_1, x_2, \dots, x_n) + \psi\left(I - \sum_{i=1}^{n} p_i x_i\right)$$

where ψ is the Lagrangian multiplier. Observing the first-order conditions,

$$\frac{\partial L_m}{\partial x_i} = \frac{\partial U}{\partial x_i} - \psi p_i = 0$$

$$\frac{\partial L_m}{\partial \psi} = I - \sum_{i=1}^{n} p_i x_i = 0$$

This gives the value of ψ to be $\frac{\frac{\partial U}{\partial x_i}}{p_i}$. For two different products p_i and p_j, the expression for ψ is used and MRS is given by

$$\frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} = \frac{p_i}{p_j}$$

Here $\frac{\partial U}{\partial x_j}$ is the marginal utility for the j-th product. Different utility functions have different forms of Marshallian demand curves. However, homogeneity is observed at zero degrees. However, Marshallian demand is related to Hicksian demand in the following way

$$h\big(\vec{p}, U(\vec{p}, U_0)\big) = x^m(\vec{p}, U_0)$$

where h is Hicksian demand, U is utility function and U_0 is the minimum limit utility. This being the fundamental demand function with Hicksian demand, brings to the end of the fundamental demand functions. Next the Slutsky equation and demand function will be an introduction to the composite demand functions.

7.3 Slutsky Demand: Named after the Russian economist Eugen Slutsky, this demand helps in analyzing the change in pattern of consumption of goods and services due to variation in the market. Mostly this variation is due to change in prices. This decomposes the Marshallian demand into the Income effect and the Substitution effect. The Income effect is the change in purchasing quantity due to the change in income of the consumer. However, the Substitution effect deals with the change in relative prices of the products where income remains constant and hence the consumption pattern of products again change. Based on these patterns, the Slutsky equation is used which deals with the effects and Hicksian demand is checked. It is given by

$$\frac{\partial x_j}{\partial p_j} = \frac{\partial h_j}{\partial p_j} - x_j\frac{\partial x_i}{\partial I}$$

Here, j-th product is considered whose Hicksian demand is given by h_j, price is given by p_j, consumption quantity is x_j and income is I. Income effect uses i-th product with consumption quantity x_i. Looking into the terms of the equation, substitution effect is given by $\frac{\partial h_j}{\partial p_j}$ and income effect is given by $x_j\frac{\partial x_i}{\partial I}$ and $\frac{\partial x_j}{\partial p_j}$ is the Slutsky effect. This helps in understanding the behavioral economics of consumers by decomposing the fundamental functions into a detailed view and take an in-depth analysis in the requirements.

VIII. HB ALGORITHM: FINDING THE OPTIMAL FINITE SET

From the outputs of EmP and UMP, the outputs that are observed often contain a bigger set of bundles which provide the maximum utility with the minimum price and satisfies Hicksian Demand for the same but in real life scenario, such a comprehensive bundle may not be the required as a whole especially when the products are alternatives of the same category of good/service. In such a case, we can further filter out the set to find an optimal solution set which contains only the very specific and highly prioritized product alternatives. In non – economic applications, such as some NP problems, shortest or least weighted path in Game Theory or even optimization problems in any domain of Engineering, this plays a very crucial role. For a specific approach, Cobb – Douglas is used for finding the utility in this case. Let there be n products numbered from 1 to n. The parameters for the i-th product will be: x_i for consumption, α_i for relative preference, p_i for price of product, I for income and U_T for target limiting utility. Now the Cobb – Douglas function for this set will be

$$U(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \qquad (8.1)$$

Using EmP, Lagrangian function for expenditure of this set, we get,

$$L_e = \sum_{i=1}^{n} p_i x_i + \omega(U_T - \prod_{i=1}^{n} x_i^{\alpha_i})$$

Where ω is the Lagrangian multiplier. Checking the first order conditions for the j-th product

$$\frac{\partial L_e}{\partial x_j} = p_j - \omega \alpha_j x_j^{\alpha_j - 1}\left(\prod_{i=1}^{n-1} x_i^{\alpha_i}\right) = 0$$

$$\frac{\partial L_e}{\partial \omega} = U_T - \left(\prod_{i=1}^{n} x_i^{\alpha_i}\right) = 0$$

which gives the value of multiplier to be

$$\omega = \frac{p_j}{\alpha_j x_j^{\alpha_j - 1}(\prod_{i=1}^{n-1} x_i^{\alpha_i})}$$

Also, the consumption quantity of the j-th product with respect to the first product, preferences of both and prices of both is given by using MRS

$$x_j = \frac{p_1 x_1 \alpha_j}{p_j \alpha_1} \qquad (8.2)$$

However, using the value of x_1 from equation 8.1, we get x_1 to be $\left(\frac{U_T}{\prod_{i=2}^{n-1} x_i^{\alpha_i}} \left(\frac{p_j \alpha_1}{p_1 \alpha_j}\right)^{\alpha_j}\right)^{\frac{1}{\alpha_j + 1}}$. Using this to find x_j we get

$$x_j = \left(\frac{p_1 \alpha_j}{p_j \alpha_1}\right)^{\frac{\alpha_j}{\alpha_j + 1}} \left(\frac{U_T}{\prod_{i=2}^{n-1} x_i^{\alpha_i}}\right)^{\frac{1}{\alpha_j + 1}} \qquad (8.3)$$

Similarly, when UMP is used, the following Lagrangian function is created

$$L_U = \prod_{i=1}^{n} x_i^{\alpha_i} + \mu(I - \sum_{i=1}^{n} p_i x_i)$$

where μ is the Lagrangian Multiplier. The first – order conditions for the j-th product give,

$$\frac{\partial L_U}{\partial x_j} = \alpha_j x_j^{\alpha_j - 1} \prod_{i=1}^{n-1} x_i^{\alpha_i} - \mu p_j = 0$$

$$\frac{\partial L_U}{\partial \mu} = I - \sum_{i=1}^{n} p_i x_i = 0$$

Now, in this condition we apply the duality of EmP and UMP and hence x_i can be written in terms of x_1 from 8.2 and write income from first – order conditions as

$$I = p_1 x_1 + \sum_{i=2}^{n} \left(\frac{p_1 \alpha_i}{p_i \alpha_1}\right) x_1$$

Hence, x_1 and a general term x_j can be derived using these terms as given in 8.4a and 8.4b. Equation 4b is declared as the HB algorithm for quantity computation of the j-th product

$$x_1 = \frac{I}{p_1 + \sum_{i=2}^{n}\left(\frac{p_1 \alpha_i}{p_i \alpha_1}\right)} \qquad (8.4a)$$

$$x_j = \frac{I\left(\frac{p_1 \alpha_j}{p_j \alpha_1}\right)}{p_1 + \sum_{i=2}^{n}\left(\frac{p_1 \alpha_i}{p_i \alpha_1}\right)} \qquad (8.4b)$$

Equating the expressions of x_j from 8.3 and 8.4b, we get the target utility to be

$$U_T = \left(\frac{I}{p_1 + \sum_{i=2}^{n}\left(\frac{p_1 \alpha_i}{p_i \alpha_1}\right)}\right)^{\alpha_j + 1} \left(\frac{p_1 \alpha_j}{\alpha_1 p_j}\right)$$

This equation is the HB equation or the HB algorithm for utility and gives the maximum utility that one can achieve when compared to the minimum expenses made. Here the term $\left(\frac{p_1\alpha_j}{\alpha_1 p_j}\right)$ is the HB constant or the HB term which gives the MRS for this function. To get the best bundle from this, equation 8.4b is the HB equation for consumption quantity and using this for the set of n elements gives the set $\{x_1^*, x_2^*, \ldots, x_n^*\}$ which gives the optimized set and this can be further filtered by Pareto Rule. Hence, sorting the set and filtering top 20% of the consumption quantities in terms of utility gives the minimum cost involved and the highest combined utility. Hence, the HB algorithm being the very first of its kind and a very helpful scenario when indifferent products are existing in the preferences.

IX. CONCLUSION

Throughout the paper a thorough discussion on different forms of functions and tools in the Consumer Theory have been discussed and finally a particular function for getting the desirable target utility which often helps to decide when the market is not known. In other words, if the consumer is new in selecting the entities, but has an idea on consumption quantities, then the HB utility algorithm can be used to find the target utility. However, if the consumption quantity is unknown, the HB consumption equation (8.4b) can be used for computing the same. This not only helps in Economics but also a crucial role in solving problems when a new material is chosen for a particular work, a function to approach a problem, a pathway to ease the Search Algorithm and even an optimization algorithm that can help in computing possible consumption of an unknown substance and the utility that one can expect from it.

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