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Inequalities between Central Location Measures and some applications

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Abstract
In this study, the measures of central tendency arithmetic, geometric, harmonic, and quadratic means were defined. The properties of these means were explained. The relationships among the arithmetic, geometric, harmonic, and quadratic means were determined. According to the inequality relationships between the means, it was found that the smallest mean is the harmonic mean, while the largest mean is the quadratic mean.
Keywords: *Mean, arithmetic, geometric, harmonic, quadratic*

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I. Introduction

Many years ago a matrix version of the arithmetic-geometric mean inequality was formulated and proved (Bhatia and Kittaneh, 1990).

A finite sequence of non-negative real numbers has an arithmetic mean that is bigger than or equal to its geometric mean, according to the traditional equality of arithmetic and geometric means (Kabluchko et al., 2018).

Inequalities for means have been the focus of extensive research recently. The literature contains a number of noteworthy inequalities for the arithmetic-geometric mean in particular (Borwein and Borwein, 1987; Bracken, 2001; Knockaert, 2003; Hayashi, 2009).

In one study, five-element vectors were obtained using the entropy value, standard deviation, arithmetic mean, geometric mean and harmonic mean of number sequences. Shannon entropy was used in the entropy calculation (Karadoğan and Karcı, 2019).

The arithmetic–geometric mean difference consistency index is a new consistency metric that is reported in a study by utilizing the difference between the arithmetic and geometric means. Using the simulation method, the acceptable consistency thresholds are established and some intriguing characteristics of the arithmetic–geometric mean difference consistency index are examined (Hu et al., 2025). The Geometric Mean Arithmetic Mean Inequality, the fundamental inequality between these means, was thoroughly examined and numerous proofs were provided in Bullen's (2003) paper. Then, other variations of this basic inequality were examined, specifically the Nanjundiah and Rado-Popoviciu type inequalities. Cebisev's inequality was discussed along with converse inequalities. The logarithmic and identric means were shown to have a few basic characteristics.

In another study, a noncommutative arithmetic-geometric mean inequality was developed with an emphasis on least means squares optimization to demonstrate that the expected convergence rate of without-replacement sampling is quicker than that of with-replacement sampling. In addition to some pathological cases, it was shown that this inequality holds for numerous classes of random matrices. The implications of this inequality for the randomized Kaczmarz algorithm for solving linear equations and stochastic gradient descent were explained in depth (Recht and Re´, 2012). In (Bhatia and Kittaneh, 2008)'s study, ideas related to matrix versions of the arithmetic–geometric mean inequality were explained.

The results of the citation modeling based on the discretized lognormal distribution show that the geometric mean is the most precise, the proportion of a nation's articles in the top 50% is nearly as precise, and the proportion of a nation's articles in the top 1% is by far the least precise, at least under the assumption that the countries compared and the overall sample have the same standard deviation parameter. The geometric mean is thus the preferred citation impact measure for comparing nations in a single field, assuming equivalent scale parameters (Thelwall, 2016).

Computer experts have been debating whether mean the arithmetic, geometric, harmonic, or perhaps the weighted harmonic is "better" for characterizing computer performance for the past 20 years. This contentious discussion has been dubbed the "war of the benchmark means" (Mashey, 2004).

In a study, the expected values and variances of new mean estimators based on the population geometric mean, population harmonic mean, and population quadratic mean of the auxiliary variable were theoretically derived. The estimators proposed using the quadratic mean were disregarded because their variances were greater than or equal to the variances of the estimators obtained using the population arithmetic mean. Nonetheless, the estimators suggested by the geometric mean and harmonic mean were examined since their variances were less than or equivalent to those of the estimators derived from the population arithmetic mean. The estimator suggested by the harmonic mean was found to have the smallest variance (Sağlam et al., 2016).

The aim of this study is to provide information about the arithmetic, geometric, harmonic, and quadratic means, to determine the relationships between them, and to demonstrate the inequalities among them.

II. Method

Four types of averages have been examined: arithmetic, geometric, harmonic, and quadratic means.

Arithmetic mean (AMe): This is the most well-known and widely used mean (to the point where it is occasionally confused with the broader term mean). It is defined as the following algebraic expression (Sýkora, 2009).

$$AMe = \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Geometric mean (GMe): Geometric mean, defined by

$$GMe = \sqrt[n]{(X_1, X_2, \dots, X_n)}$$

Harmonic mean (HMe): The harmonic mean is an important metric of central tendency that is used more frequently when averaging speeds over the same distances. The harmonic mean is calculated by taking the reciprocals of the numbers that were averaged. It is the reciprocal of the arithmetic mean, which is the reciprocal of the averaged numbers (Ogbi, 2012).

$$HMe = \frac{n}{\frac{1}{X_1} + \frac{1}{X_2} + \dots + \frac{1}{X_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

Quadratic mean (QMe): The quadratic mean formula, that is, the root mean square formula, can be used to show the magnitude of a given set of numbers.

$$QMe = \sqrt{\frac{X_1^2 + X_2^2 + \dots + X_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{n}}$$

When the number of observations is 2, the means are calculated as follows.

$$AMe = \frac{X_1 + X_2}{2}$$

$$GMe = \sqrt{X_1 . X_2}$$

$$HMe = \frac{2}{\frac{1}{X_1} + \frac{1}{X_2}} = \frac{2X_1X_2}{X_1 + X_2}$$

$$QMe = \sqrt{\frac{X_1^2 + X_2^2}{2}}$$

When the number of observations is 3, the means are calculated as follows.

$$AMe = \frac{X_1 + X_2 + X_3}{3}$$

$$GMe = \sqrt[3]{X_1 . X_2 . X_3}$$

$$HMe = \frac{3}{\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3}}$$

$$QMe = \sqrt{\frac{X_1^2 + X_2^2 + X_3^2}{3}}$$

III. Results

According to the arithmetic–geometric mean inequality, the arithmetic mean is greater than or equal to the geometric mean for any series of n non-negative real integers, X_1, X_2, ..., X_n:

$$\frac{X_1 + X_2 + \dots + X_n}{n} \geq X_1 X_2 \dots X_n^{(\frac{1}{n})}$$

One way to think of this is as a special example of Maclaurin's inequality (m = n) (Israel et al., 2016).
For the number of observations n=2, that is, for (X_1, X_2), the means and the inequalities among the means are summarized as follows. The arithmetic mean–geometric mean inequality is given by:

$$GMe \leq AMe$$
$$\sqrt{X_1.X_2} \leq \frac{X_1 + X_2}{2}$$

By squaring both sides of the inequality, the square root is eliminated.

$$\left(\sqrt{X_1.X_2}\right)^2 \leq \left(\frac{X_1 + X_2}{2}\right)^2$$
$$X_1.X_2 \leq \frac{(X_1^2 + 2X_1X_2 + X_2^2)}{4}$$
$$4X_1X_2 \leq (X_1^2 + 2X_1X_2 + X_2^2)$$
$$0 \leq (X_1^2 - 4X_1X_2 + 2X_1X_2 + X_2^2)$$
$$0 \leq (X_1^2 - 2X_1X_2 + X_2^2)$$
$$0 \leq (X_1 - X_2)^2$$

That is,

$$(X_1 - X_2)^2 \geq 0$$

The square of any expression can never be negative. Therefore, this inequality holds. The geometric mean–harmonic mean inequality is given as follows.

$$HMe \leq GMe$$
$$\frac{2X_1X_2}{X_1 + X_2} \leq \sqrt{X_1X_2}$$

Both sides of the inequality are squared.

$$\left(\frac{2X_1X_2}{X_1 + X_2}\right)^2 \leq \left(\sqrt{X_1X_2}\right)^2$$
$$\frac{4X_1^2X_2^2}{X_1^2 + 2X_1X_2 + X_2^2} \leq X_1X_2$$

Both sides of the inequality can be simplified by X_1X_2. In this case,

$$\frac{4X_1X_2}{X_1^2 + 2X_1X_2 + X_2^2} \leq 1$$

is obtained. From this,

$$4X_1X_2 \leq X_1^2 + 2X_1X_2 + X_2^2$$
$$0 \leq X_1^2 - 4X_1X_2 + 2X_1X_2 + X_2^2$$
$$0 \leq X_1^2 - 2X_1X_2 + X_2^2$$
$$0 \leq (X_1 - X_2)^2$$

Namely,
The inequality

$$(X_1 - X_2)^2 \geq 0$$

holds.
When there are two observations, the inequality between the arithmetic mean and the quadratic mean is as follows.

$$AMe \leq QMe$$
$$\frac{X_1 + X_2}{2} \leq \sqrt{\frac{X_1^2 + X_2^2}{2}}$$

Both sides of the inequality are squared.

$$\left(\frac{X_1 + X_2}{2}\right)^2 \leq \left(\sqrt{\frac{X_1^2 + X_2^2}{2}}\right)^2$$
$$\frac{(X_1^2 + 2X_1X_2 + X_2^2)}{4} \leq \frac{X_1^2 + X_2^2}{2}$$

For the sake of convenience, both sides of the inequality are multiplied by 4.

$$(X_1^2 + 2X_1X_2 + X_2^2) \leq 2(X_1^2 + X_2^2)$$
$$(X_1^2 + 2X_1X_2 + X_2^2) \leq 2X_1^2 + 2X_2^2$$
$$2X_1X_2 \leq 2X_1^2 + 2X_2^2 - X_1^2 - X_2^2$$
$$2X_1X_2 \leq X_1^2 + X_2^2$$
$$0 \leq X_1^2 + X_2^2 - 2X_1X_2$$

$$0 \leq (X_1 - X_2)^2$$

That is, the inequality $(X_1 - X_2)^2 \geq 0$ is satisfied.

For the number of observations n=3, that is, X_1, X_2, X_3, the means and the inequalities between the means are summarized as follows.

Inequality between the arithmetic mean and the geometric mean

$$GMe \leq AMe$$

$$\sqrt[3]{X_1.X_2.X_3} \leq \frac{X_1 + X_2 + X_3}{3}$$

To eliminate the roots in this expression, both sides of the inequality are raised to the power of three (cubed).

$$\left(\sqrt[3]{X_1.X_2.X_3}\right)^3 \leq \left(\frac{X_1 + X_2 + X_3}{3}\right)^3$$

$$\left(\sqrt[3]{X_1.X_2.X_3}\right)^3 = X_1.X_2.X_3$$

$$\left(\frac{X_1 + X_2 + X_3}{3}\right)^3 = \frac{X_1^3 + X_2^3 + X_3^3 + 3X_1^2X_2 + 3X_1^2X_3 + 3X_2^2X_1 + 3X_2^2X_3 + 3X_3^2X_1 + 3X_3^2X_2 + 6X_1X_2X_3}{27}$$

It is as follows (Anonymous, 2025a).

$$X_1.X_2.X_3 \leq \frac{X_1^3 + X_2^3 + X_3^3 + 3X_1^2X_2 + 3X_1^2X_3 + 3X_2^2X_1 + 3X_2^2X_3 + 3X_3^2X_1 + 3X_3^2X_2 + 6X_1X_2X_3}{27}$$

$$27X_1X_2X_3 \leq X_1^3 + X_2^3 + X_3^3 + 3X_1^2X_2 + 3X_1^2X_3 + 3X_2^2X_1 + 3X_2^2X_3 + 3X_3^2X_1 + 3X_3^2X_2 + 6X_1X_2X_3$$

$$21X_1X_2X_3 \leq X_1^3 + X_2^3 + X_3^3 + 3X_1^2X_2 + 3X_1^2X_3 + 3X_2^2X_1 + 3X_2^2X_3 + 3X_3^2X_1 + 3X_3^2X_2$$

$$21X_1X_2X_3 \leq (X_1 + X_2 + X_3)^3 - 6X_1X_2X_3$$

It is obtained. If $X_1 = 1, X_2 = 2,\ and\ X_3 = 3$ are substituted into this inequality, the inequality

$$21\ (1 * 2 * 3) \leq (1 + 2 + 3)^3 - 6(1 * 2 * 3)$$

$$126 \leq 216 - 36$$

$$126 \leq 180$$

holds.

Inequality between the arithmetic mean and the quadratic mean

$$AMe \leq QMe$$

$$\frac{X_1 + X_2 + X_3}{3} \leq \sqrt{\frac{X_1^2 + X_2^2 + X_3^2}{3}}$$

To eliminate the root, both sides of the inequality are squared.

$$\left(\frac{X_1 + X_2 + X_3}{3}\right)^2 \leq \left(\sqrt{\frac{X_1^2 + X_2^2 + X_3^2}{3}}\right)^2$$

$$\left(\frac{X_1 + X_2 + X_3}{3}\right)^2 \leq \frac{X_1^2 + X_2^2 + X_3^2}{3}$$

$$\frac{X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_1X_3 + 2X_2X_3}{9} \leq \frac{X_1^2 + X_2^2 + X_3^2}{3}$$

$$X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_1X_3 + 2X_2X_3 \leq 3(X_1^2 + X_2^2 + X_3^2)$$

$$X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_1X_3 + 2X_2X_3 \leq 3X_1^2 + 3X_2^2 + 3X_3^2$$

$$2X_1X_2 + 2X_1X_3 + 2X_2X_3 \leq 2X_1^2 + 2X_2^2 + 2X_3^2$$

Both sides of the inequality are divided by 2.

$$X_1X_2 + X_1X_3 + X_2X_3 \leq X_1^2 + X_2^2 + X_3^2$$

When $X_1 = 1, X_2 = 2,\ and\ X_3 = 3$ are substituted into this inequality, it is found that

$$1 * 2 + 1 * 3 + 2 * 3 \leq 1^2 + 2^2 + 3^2$$

$$11 \leq 14$$

and the inequality holds.

Inequality between the harmonic mean and the geometric mean

$$HMe \leq GMe$$

Again, for n=3 observations, that is, X_1, X_2, X_3, the inequality holds.

$$\frac{3}{\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}} \leq \sqrt[3]{X_1.X_2.X_3}$$

To eliminate the root in the inequality, both sides are raised to the power of three (cubed).

$$\left(\frac{3}{\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}}\right)^3 \leq \left(\sqrt[3]{X_1.X_2.X_3}\right)^3$$

First, the expression

$$\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}\right)^3$$

is expanded. All terms are written over a common denominator of $X_1X_2X_3$. Then,

$$\left(\frac{X_2X_3+X_1X_3+X_1X_2}{X_1X_2X_3}\right)^3$$

To raise a fraction to a power, the numerator and denominator are each raised to that power separately.

$$\left(\frac{X_2X_3+X_1X_3+X_1X_2}{X_1^3X_2^3X_3^3}\right)^3$$

The three-term expression is expanded using the formula.

$$\frac{X_2^3X_3^3+X_1^3X_3^3+X_1^3X_2^3+3X_1X_2^2X_3^3+3X_1X_2^3X_3^2+3X_1^2X_2X_3^3+3X_1^3X_2X_3^2+3X_1^2X_2^3X_3+3X_1^3X_2^2X_3+6X_1^2X_2^2X_3^2}{X_1^3X_2^3X_3^3}$$

It is obtained. When this expression is arranged according to the formula for the cube (third power) of the harmonic mean, when the

$$\left(\frac{3}{\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}}\right)^3$$

operation is performed. Thus

$$\frac{27X_1^3X_2^3X_3^3}{X_2^3X_3^3+X_1^3X_3^3+X_1^3X_2^3+3X_1X_2^2X_3^3+3X_1X_2^3X_3^2+3X_1^2X_2X_3^3+3X_1^3X_2X_3^2+3X_1^2X_2^3X_3+3X_1^3X_2^2X_3+6X_1^2X_2^2X_3^2}$$

is obtained (Anonymous, 2025b)

$$\left(\sqrt[3]{X_1.X_2.X_3}\right)^3 = X_1X_2X_3$$

$$\frac{27X_1^3X_2^3X_3^3}{X_2^3X_3^3+X_1^3X_3^3+X_1^3X_2^3+3X_1X_2^2X_3^3+3X_1X_2^3X_3^2+3X_1^2X_2X_3^3+3X_1^3X_2X_3^2+3X_1^2X_2^3X_3+3X_1^3X_2^2X_3+6X_1^2X_2^2X_3^2} \leq X_1X_2X_3$$

The inequality holds. When both sides of this inequality are simplified by the expression $X_1X_2X_3$

$$\frac{27X_1^2X_2^2X_3^2}{X_2^3X_3^3+X_1^3X_3^3+X_1^3X_2^3+3X_1X_2^2X_3^3+3X_1X_2^3X_3^2+3X_1^2X_2X_3^3+3X_1^3X_2X_3^2+3X_1^2X_2^3X_3+3X_1^3X_2^2X_3+6X_1^2X_2^2X_3^2} \leq 1$$

It is obtained. When $X_1 = 1, X_2 = 2, X_3 = 3$ are substituted into this inequality,

$$\frac{972}{216+27+8+324+216+162+54+72+36+216} \leq 1$$

$$\frac{972}{1331} \leq 1$$

$$0.730 \leq 1$$

It is found, and the inequality holds. In line with this information, the following inequality result has emerged.

$$HMe \leq GMe \leq AMe \leq QMe$$

IV. Conclusions

It is evident that when all observation values are equal, the arithmetic, geometric, harmonic, and quadratic means will also be equal. If it is known that the observation values differ for each mean calculation, it has been observed that all mean values are also different. In this case, the inequality HMe ≤ GMe ≤ AMe ≤ QMe holds. This result is extremely important for determining which type of mean should be used in different situations.

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