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Matrix Games with Intuitionistic Fuzzy Goals

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Abstract—In this paper, we present an application of intuitionistic fuzzy programming to two person matrix games for the solution with mixed strategies. We use linear membership and non membership functions for such computation. We introduce the intuitionistic fuzzy (IF) goal for the choice of a strategy in a pay-off matrix in order to incorporate ambiguity of human judgments and a player wants to optimize his/her degree of attainment of the IF goal. It is shown that this is the optimal solution of the mathematical programming problem. In addition, numerical example is also presented to illustrate the methodology.

Keywords—Matrix game, Intuitionistic fuzzy sets, Fuzzy optimization, Intuitionistic fuzzy optimization, Fuzzy goal

I. INTRODUCTION

Game theory [6, 7] is a mathematical tool for the analysis of conflicting interests' situations, which includes players or decision makers (DM) who select various strategies from a set of available strategies. Fuzziness in game theory was studied by various researchers [1, 5, 10, 11, 16], where the goals are viewed as fuzzy sets, but they are very limited and in many cases they do not represent exactly the real problems. In practical situation, due to insufficiency in the information available, it is not easy to describe the fuzzy constraint conditions by ordinary fuzzy sets and consequently, the evaluation of membership values is not always possible up to DM's 1 satisfaction. Due to the same reason evaluation of non-membership values is not always possible and consequently there remains an indeterministic part of which hesitation survives. In such situation intuitionistic fuzzy set (IFS), Atanassov [8] serve better our required purpose. Intuitionistic fuzziness in matrix games can appear in so many ways, but two cases of fuzziness seem to be very natural. These being the one in which DMs have IF goals and the other in which the elements of the pay-off matrix are given by Intuitionistic fuzzy numbers. These two classes of fuzzy matrix games are referred as matrix games with IF goals and matrix games with IF pay-offs [12]. No studies, however, have been made for solution of matrix game with IF goal. We introduce here a new approach to solve matrix game with IF goal. We assume that each player has a IF goal for the choice of the strategy. IF goal for the strategies in a pay-off matrix has been formulated in order to ambiguity of human judgment. We assume that, DMs want to optimize the degree of attainment of the IF goal.

For the purpose, this paper is organized as follows: In section 2, we shall give some basic definitions and notations on IFS. In section 3, we shall determine the solution of the matrix game on the basis of defining degree of attainment of the IF goal.

II. INTUITIONISTIC FUZZY SETS

Before we proceed further in defining the bi-matrix games with IF goal, we introduce first some relevant basic preliminaries, notations and definitions of IFS, in particular the works of Atanassov [8].

Definition 1 An *intuitionistic fuzzy set (IFS)* in a given universal set *U* is an expression *A* given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\} \qquad (1)$$

where $\mu_A(x) : U \to [0,1]$ and $\nu_A(x) : U \to [0,1]$ satisfy the following conditions:

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \forall x \in U$$

which is known as intuitionistic condition. The number $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership respectively of the element $x(\in U)$ to the set *A*. The set of all IFS on *U* is denoted by *IFS*(*U*).

Definition 2 For all $A \in IFS(U)$, the intuitionistic index of the element *x* in the set *A* is $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ which is also called the degree of uncertainty or in deterministic part of *x* satisfying $0 \le \pi_A(x) \le 1$ for all $x \in U$.

Obviously, when $\pi_A(x) = 0, \forall x \in U$, i.e, $\mu_A(x) + \nu_A(x) = 1$, the set *A* is a fuzzy set as follows:

$A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in U\} = \{\langle x, \mu_A(x)\rangle : x \in U\}.$
Therefore fuzzy set is a especial IFS.

2.1 Operations on IFSs

For the sake of completeness, we first recall the definitions of some operations. Let *A* and *B* be two IFSs of the set *U*, then their operations are defined by membership functions

(i) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle : x \in U\}$

(ii) $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle : x \in U\}$.

The degree of acceptance $\mu_A(x)$ and of nonacceptance $\nu_A(x)$ can be arbitrary.

Note: Thus the numbers $\mu_A(x)$ and $\nu_A(x)$ reflect respectively the extent of acceptance and the degrees of rejection of the element *x* to the set *A*, and the numbers $\pi_A(x)$ is the extent of indeterminacy between both.

2.2 Fuzzy optimization model

In fuzzy optimization problem (fuzzy mathematical programming, fuzzy optimal control, etc.),the objective(s) and/or constraints and relations are represented by fuzzy sets. These fuzzy sets explain the degree of satisfaction of the respective condition and are expressed by their membership function. Let us consider an optimization problem

$$f_i(x) \to \min;\ i = 1, 2, \cdots, p$$

$$\text{subject to,} \qquad g_j(x) \le 0;\ j = 1, 2, \cdots, q,$$

where *x* denotes the unknowns, $f_i(x)$ denotes the objective functions, $g_j(x)$ denotes constraints (non-equalities), *p* denotes the number of objectives and *q* denotes the number of constraints. The solution of this crisp optimization problem satisfies all constraints exactly. In the analogous, fuzzy optimization problem the degree of satisfaction of the objective(s) and the constraints is maximized:

$$f_i(x) \to \widetilde{\min};\ i = 1, 2, \cdots, p$$

$$\text{subject to,} \qquad g_j(x) \,\tilde{\le}\, 0;\ j = 1, 2, \cdots, q,$$

where $\widetilde{\min}$ denotes fuzzy minimization and $\tilde{\le}$ denotes fuzzy inequality. It is transferred via Bellman-Zadeh's approach to the following optimization problem: To maximize the degree of membership of the objective(s) and constraints to the respective fuzzy sets:

$$\max\ \mu_i(x);\quad x \in \Re^n;\quad i = 1, 2, \cdots, p+q$$

$$\text{subject to} \qquad 0 \le \mu_A(x) \le 1,$$

where $\mu_i(x)$ denotes degree of acceptance of *x* to the respective fuzzy sets. When the degree of rejection(non-membership) is defined simultaneously with the degree of acceptance(membership) and when both theses degree are not complementary each other, then IFS can be used as a more general and full tool for describing this uncertainty optimization model.

2.3 Intuitionistic fuzzy optimization model

Generally, an optimization problem includes objective(s) and constraints. Intuitionistic fuzzy optimization (IFO), a method of uncertainty optimization, is put forward on the basis of IFS [8]. According to IFO theory [15], we are to maximize the degree of acceptance of the IF objective(s) and constraints and to minimize the degree of rejection of IF objective(s) and constraints as

$$\max_x \{\mu_k(x)\};\quad x \in \Re^n;\quad k = 1, 2, \ldots, 2n$$

$$\min_{x} \{ \nu_k(x) \}; \qquad k = 1, 2, \ldots, 2n$$

$$\mu_k(x),\ \nu_k(x) \ \geq 0 \ ; k = 1, 2, \ldots, 2n$$

$$\mu_k(x) \ \geq \ \nu_k(x) \ ; k = 1, 2, \ldots, 2n$$

$$0 \leq \mu_A(x) + \nu_A(x) \ \leq 1; k = 1, 2, \ldots, 2n$$

where $\mu_k(x)$ denotes the degree of acceptance of *x* from the *k*th IFS and $\nu_k(x)$ denotes the degree of rejection of *x* from the *k*th IFS. These IFS include IF objective (s) and constraints. It is an extension of fuzzy optimization in which the degrees of rejection of objective(s) and constraints are considered together with the degrees of satisfaction. According to Atanassov property *2.1(iv)* (in this paper) of IFS, the conjunction of intuitionistic fuzzy objective(s) and constraints is defined as

$$A \cap B = \{\langle x, \min\{ \mu_A(x), \mu_B(x)\}, \max\{ \nu_A(x), \nu_B(x)\}\rangle : x \in U\} \tag{2}$$

which is defined as the intuitionistic fuzzy decision set (IFDS), where *A* denotes the integrated intuitionistic fuzzy objective and *B* denotes integrated intuitionistic fuzzy constraint set and they can be written as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\} = \bigcap_{i=1}^{n} A^{(i)} = \{\langle x, \min_{i=1}^{n} \mu_i(x), \max_{i=1}^{n} \nu_i(x)\rangle : x \in U\}. \tag{3}$$

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\} = \bigcap_{j=1}^{n} B^{(j)} = \{\langle x, \min_{j=1}^{n} \mu_i(x), \max_{j=1}^{n} \nu_i(x)\rangle : x \in U\}. \tag{4}$$

Let the IFDS (2) be denoted by C, then

$$C = A \cap B = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}, \tag{5}$$

where, $$\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\} = \min_{k=1}^{n+n} \mu_k(x) \tag{6}$$

$$\nu_C(x) = \max\{\nu_A(x), \nu_B(x)\} = \max_{k=1}^{n+n} \nu_k(x), \tag{7}$$

where $\mu_C(x)$ denotes the degree of acceptance of IFDS and $\nu_C(x)$ denotes the degree of rejection of IFDS. The formula can be transformed to the following system of inequalities

$$\max \alpha \ , \min \ \beta$$

$$\alpha \leq \mu_k(x); \qquad k = 1,2,\ldots,2n$$

$$\beta \geq \nu_k(x); \qquad k = 1,2,\ldots,2n$$

$$\alpha \geq \beta; \qquad and \ \ \alpha + \beta \leq 1; \ \ \alpha, \beta \geq 0, \tag{8}$$

where α denotes the minimal acceptable degree of objective(s) and constraints and β denotes the maximal degree of rejection of objective(s) and constraints. The IFO model can be changed into the following certainty (non-fuzzy) optimization model as:

$$\max\ (\alpha - \beta)$$

$$\alpha \leq \mu_k(x); \qquad k = 1,2,\ldots,2n$$

$$\beta \geq \nu_k(x); \qquad k = 1,2,\ldots,2n$$

$$\alpha \geq \beta; \qquad and \ \ \alpha + \beta \leq 1; \ \ \alpha, \beta \geq 0 \tag{9}$$

which can be easily solved by some simplex methods.

III. MATHEMATICAL MODEL OF A MATRIX GAME

Let $i \in \{1, 2, \ldots, m\}$ be a pure strategy available for player I and $j \in \{1, 2, \ldots, n\}$ be a pure strategy available for player II. When player I chooses a pure strategy *i* and the player II chooses a pure strategy *j*, then a_{ij} is a payoff for player I and $-a_{ij}$ be a payoff for player II. The two person zero sum matrix game can be represented as a pay-off matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

3.1 Game without saddle point

There are games having no saddle points. Consider a simple *2 × 2* game with no saddle point with the pay-off matrix

$$\begin{array}{c} \quad B_1 \quad B_2 \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}.$$

where, $\max_i\{\min_j \; a_{ij}\} \neq \min_j\{\max_i \; a_{ij}\}$. To solve such IG, Neumann [6] has introduced the concept of mixed strategy in classical form. We denote the sets of all mixed strategies, called strategy spaces, available for players I, II by

$$S_I = \{(x_1, x_2, ..., x_m) \in \Re_+^m;\; x_i \geq 0;\; i = 1,2,..., m \;\; and \;\; \sum_{i=1}^{m} x_i = 1\}$$

$$S_{II} = \{(y_1, y_2, ..., y_n) \in \Re_+^n;\; y_i \geq 0;\; i = 1,2,..., n \;\; and \;\; \sum_{i=1}^{n} y_i = 1\},$$

where, $\Re_+^m$ denotes the *m*−dimensional non negative Euclidean space. Since the player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives available to him/her or a mixed strategy in terms of game theory. We shall denote this matrix game by

$$\Gamma_1 = \langle \{I, II\}, S_I \times S_{II}, A \rangle.$$

The gain function $V : S_I \times S_{II} \to \Re,$, the set of all real numbers, is called two person game in mixed strategies.

Definition 3 (Expected payoff) : If the mixed strategies *x* and *y* are proposed by the player I and player II respectively, then the expected pay-off of the player I when the player II uses the strategy y is defined by

$$E(x, y) = x^T A y = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j \qquad (10)$$

The existing theory of crisp games has certain limitations because of uncertainties and ambiguous communication. The purpose of this paper is to obviate such difficulties. Now, we define the meaning of an IF goal and try to explain how the players will play the game in an IF environment.

3.2 Matrix game with IF Goal

The IF goal models are described on the basis of maxmin and minmax principles of crisp matrix game theory. First, we define some terms which are useful in the solution procedure. Let the domain for the player I be defined by

$$D = \{x^T A y : (x, y) \in S_I \times S_{II} \subset \Re^m \times \Re^n\} \subseteq \Re. \qquad (11)$$

Definition 4 (IF goal): A IF goal G_1 for player I is defined as a IFS on *D* characterized by the membership and non membership functions

$$\mu_{G_1} : D \to [0,1] \text{ and } \nu_{G_1} : D \to [0,1]$$

$$\text{or simply, } \mu_1 : D \to [0,1] \text{ and } \nu_1 : D \to [0,1]$$

such that $0 \leq \mu_1(x) + \nu_1(x) \leq 1$. Similarly, a IF goal for player II is IFS on *D* characterized by the membership function $\mu_{G_2} : D \to [0,1]$ and non membership function $\nu_{G_2} : D \to [0,1]$ such that $0 \leq \mu_2(x) + \nu_2(x) \leq 1$.

A membership, non-membership function value for a IF goal can be interpreted as the degree of attainment [5] of the IF goal for a strategy of a payoff. According to Atanassov's property [8] of IFS (seen in section 2.1 in this paper), the intersection of IF objective(s) and constraints is defined as

$$C = A \cap B = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in X\},$$

where, $\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}; \ \nu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$

where *A* denotes integrated IF objective and *B* denotes integrated IF constraint set. $\mu_C(x)$ denotes the degree of acceptance of IF decision set and $\nu_C(x)$ denotes the degree of rejection of IF decision set.

Definition 5 (Degree of attainment of IF goal): For any pair of mixed $(x, y) \in S_I \times S_{II}$, the degree of attainment of the IF goal for player I is defined by the membership and non membership functions as

$$\max_{x \in S_I}\{\mu_1(x^T Ay)\} \text{ and } \min_{x \in S_I}\{\nu_1(x^T Ay)\}.$$

The degree of attainment of the IF goal can be considered to be a concept of a degree of satisfaction of the fuzzy decision [8, 9], when the IF constraint can be replaced by expected pay-off. Let player I supposes that, player II will choose a strategy y so as to minimize player I's membership function μ_1 and non-membership function ν_1. Let us assume that, a player has no information about his opponent or the information is not useful for the decision making if he/she has. Let player I chooses $x \in S_1$, then the least degree of μ_1 and greatest degree of ν_1 for his goal will be

$$v(x) = \min_{y \in S_{II}}\{\mu_1(x^T Ay)\} \text{ and } \max_{x \in S_{II}}\{\nu_1(x^T Ay)\}. \tag{12}$$

Here player I chooses a strategy so as to maximize μ_1 and minimize ν_1 of the IF goal $v(x)$. Similar for player II. Thus, when a player has two different strategies, he/she prefers the strategy possessing the higher membership function value and lower non membership function value in comparison to the other.

Definition 6 (Maxmin Value): For any pair of mixed strategies $(x, y) \in S_I \times S_{II}$, the maxmin value with respect to a degree of attainment of the IF goal for player I is defined as

$$\max_{x \in S_I}\min_{y \in S_{II}}\{\mu_1(x^T Ay)\} \text{ and } \min_{x \in S_I}\max_{y \in S_{II}}\{\nu_1(x^T Ay)\}. \tag{13}$$

Similarly, the minmax value with respect to a degree of attainment of the IF goal for player II is defined as

$$\max_{y \in S_{II}}\min_{x \in S_I}\{\mu_2(x^T Ay)\} \text{ and } \max_{x \in S_I}\min_{y \in S_{II}}\{\nu_2(x^T Ay)\}. \tag{14}$$

Thus the player I wishes to determine $x^* \in S_I$ (respectively $y^* \in S_{II}$) such that the maxmin value with respect to the degree of attainment of the IF goal for player I is attained. Similarly, for player II. For this, we assume that membership and nonmembership functions { μ_k, ν_k ; *k =1, 2*} for player I and player II respectively are piecewise linear.

Definition 7 (Solution of IF matrix game): Let λ_1, λ_2 be scalars representing the aspiration levels of player I and player II respectively. A pair $(x^*, y^*) \in S_I \times S_{II}$ is called a solution of the IF matrix game if

$$\mu_1(x^{*T} Ay) \geq \lambda_1 \text{ and } \nu_1(x^{*T} Ay) \leq \lambda_1'; \ \forall\, y \in S_{II} \tag{15}$$

where, '$\geq$' and '$\leq$' are IF versions greater than and less than respectively (seen in section 2.2 in this paper) by Atanassov [8]. If (λ_1, λ_2) is the reasonable solution of IF game then λ_1 (respectively λ_2) is called a reasonable value of the player I (respectively player II). x is called an optimal strategy for player I and y is called an optimal strategy for player II.

We now analyze the optimization problems for player I and player II so as to obtain a solution of the given matrix game with respect to the degree of attainment of the IF goal. By using the above definitions for the IF game, we construct the following IF linear programming problem (LPP) for player I and II.

3.3 Optimization problem for player I

The linear membership and non membership functions [8] of the fuzzy goal $\mu_1(x^T Ay)$ and $\nu_1(x^T Ay)$, for the player I can mathematically be represented as :

$$\mu_1(x^TAy) = \begin{cases} 1; & x^TAy \geq \overline{a} \\ \dfrac{x^TAy - \underline{a}}{\overline{a} - \underline{a}} & ;\ \underline{a} < x^TAy < \overline{a} \\ 0; & x^TAy \leq \underline{a} \end{cases}$$

$$\nu_1(x^TAy) = \begin{cases} 1; & x^TAy \leq \underline{a} \\ \dfrac{-x^TAy + \overline{a}}{\overline{a} - \underline{a}} & ;\ \underline{a} < x^TAy < \overline{a} \\ 0; & x^TAy \geq \overline{a} \end{cases}$$

where $\underline{a}$ and $\overline{a}$ are the tolerances of the expected pay-off x^TAy and $\mu_1(x^TAy)$ should be determined in objective allowable region [$\underline{a}$, $\overline{a}$]. For player I, $\underline{a}$ and $\overline{a}$ are the pay-off giving the worst and the best degree of satisfaction respectively. Although $\underline{a}$ and $\overline{a}$ would be any scalars with $\underline{a}$ < $\overline{a}$, Nishizaki [5] suggested that, parameters $\underline{a}$ and $\overline{a}$ can be taken as

$$\overline{a} = \max_x \max_y x^TAy = \max_i \max_j a_{ij}$$

$$\underline{a} = \min_x \min_y x^TAy = \min_i \min_j a_{ij} .$$

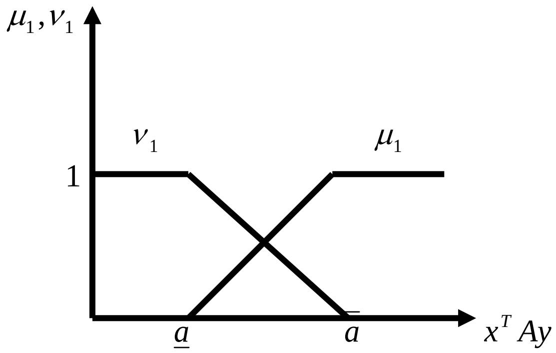

Figure 1

Thus player I is not satisfied by the pay-off less than $\underline{a}$ but is fully satisfied by the pay-off greater than $\overline{a}$. Thus, the conditions $\underline{a} \leq \min_{i,j} a_{ij}$ and $\overline{a} \geq \max_{i,j} a_{ij}$ hold.

Theorem 1: If all the membership and non-membership functions of the IF goal for player I are linear, a maxmin solution with respect to the degree of attainment of the aggremented IF goal can be obtained by solving the following certainty mathematical problem :

P1:

$$\left.\begin{aligned} &\max\ \alpha \\ s.t.\quad &\sum_{i=1}^{m} \frac{a_{ij}}{\overline{a} - \underline{a}} x_i - \frac{\underline{a}}{\overline{a} - \underline{a}} \geq \alpha; \quad j = 1,2,\ldots,n \\ &e^T x = 1; \quad x \geq 0 \end{aligned}\right\} \tag{16}$$

where α denotes the maximum acceptance degree of constraints fixed by the player I. Similarly,

P2:

$$\left.\begin{array}{l} \min\ \beta \\ s.t. \quad \sum_{i=1}^{m} \frac{a_{ij}}{\bar{a}-\underline{a}} x_i - \frac{\bar{a}}{\bar{a}-\underline{a}} \le \beta;\ j=1,2,...,n \\ \alpha \ge \beta;\ \ \alpha+\beta \le 1;\ \ \alpha,\beta \ge 0 \end{array}\right\} \quad (17)$$

where β denotes the minimal rejection degree of constraints fixed by the player I.

Proof : Let $a_{ij}^{*} = \frac{a_{ij}}{\bar{a}-\underline{a}}$ and $c_1 = -\frac{\underline{a}}{\bar{a}-\underline{a}}$, $c_2 = -\frac{\bar{a}}{\bar{a}-\underline{a}}$. The maximum problem for player I is

$\max_{x\in S_I} \min_{y\in S_{II}} \{\mu_1(x^T Ay)\}$ and $\min_{x\in S_I} \max_{y\in S_{II}} \{\nu_1(x^T Ay)\}$, which can be transformed into

$$\max_{x\in S_I} \min_{y\in S_{II}} \left[\frac{x^T Ay-\underline{a}}{\bar{a}-\underline{a}}\right] = \max_{x\in S_I} \min_{y\in S_{II}} \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{*} x_i y_j + c_1\right]$$
$$= \max_{x\in S_I} \min_{j} \left[\sum_{i=1}^{m} a_{ij}^{*} x_i + c_1\right]$$

and

$$\min_{x\in S_I} \max_{y\in S_{II}} \left[\frac{x^T Ay-\bar{a}}{\bar{a}-\underline{a}}\right] = \min_{x\in S_I} \max_{y\in S_{II}} \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{*} x_i y_j + c_2\right]$$
$$= \min_{x\in S_I} \max_{j} \left[\sum_{i=1}^{m} a_{ij}^{*} x_i + c_2\right].$$

Thus, taking, $\min_{j}\left[\sum_{i=1}^{m} a_{ij}^{*} x_i + c_1\right] = \lambda_1$, $\max_{j}\left[\sum_{i=1}^{m} a_{ij}^{*} x_i + c_2\right] = \lambda_1'$, the maximum problem for player I reduces to the LPP (16). Similarly, we can get the desired LPP (17). From (16) and (17) we see that the constraints are separable in the decision variable *x*, thus these two models (16) and (17) can be changed into the following non-fuzzy optimization model.

Theorem 2 If all the membership and non-membership functions of the IF goal are linear, a solution of the game with respect to the degree of attainment of the IF goal aggregated by minimum component is equal to the optimal solution of the LPP

$$\left.\begin{array}{l} \max(\alpha-\beta+\lambda_1) \\ \sum_{i=1}^{m} \frac{a_{ij}}{\bar{a}-\underline{a}} x_i - \frac{\underline{a}}{\bar{a}-\underline{a}} \ge \alpha;\ \ j=1,2,\ldots,n \\ \sum_{i=1}^{m} \frac{a_{ij}}{\bar{a}-\underline{a}} x_i - \frac{\bar{a}}{\bar{a}-\underline{a}} \le \beta;\ \ j=1,2,\ldots,n \\ \left[\sum_{i=1}^{m} a_{ij} x_i + c_1\right] \le \lambda_1;\ \ j=1,2,\ldots,n \\ \left[\sum_{i=1}^{m} a_{ij} x_i + c_2\right] \ge \lambda_1';\ \ j=1,2,\ldots,n \\ \alpha \ge \beta;\ \ \alpha+\beta \le 1;\ \ \alpha,\beta \ge 0 \\ e^T x = 1;\quad x \ge 0. \end{array}\right\} \quad (18)$$

If all the membership and non-membership functions of the IF goals are linear, then the optimal solution of (18) is equal to degree of attainment of the fuzzy goal for the matrix game. Since S_I is convex polytopes, for the choice of linear membership and non-membership functions, the existence of solution of the game is guaranteed from the equation (18). Also if (x^*, α^*, β^*) is an optimal solution of (18), then x^* is an optimal strategy for player I and α^*, β^* is the degree to which the aspiration level $\lambda_1, \lambda_1^{'}$ of player I can be met by choosing to play the strategy x^*.

3.4 Optimization problem for player II

Here we are consider player II maxmin solution with respect to the degree of attainment of his IF goal. The membership and nonmembership functions of the IF goal $\mu_2(x^T Ay)$ and $\nu_2(x^T Ay)$, can be represented as :

$$\mu_2(x^T Ay) = \begin{cases} 0; & x^T Ay \ge \bar{a} \\ \dfrac{\bar{a} - x^T Ay}{\bar{a} - \underline{a}} & ; \ \underline{a} < x^T Ay < \bar{a} \\ 1; & x^T Ay \le \underline{a} \end{cases}$$

$$\nu_2(x^T Ay) = \begin{cases} 0; & x^T Ay \le \underline{a} \\ \dfrac{\underline{a} - x^T Ay}{\underline{a} - \bar{a}} & ; \ \underline{a} < x^T Ay < \bar{a} \\ 1; & x^T Ay \ge \bar{a} \end{cases}$$

where $\underline{a}$ and $\bar{a}$ are the tolerances of the expected pay-off $x^T Ay$ and $\mu_2(x^T Ay)$ should be determined in objective allowable region [$\underline{a}$, $\bar{a}$].

Theorem 3 If all the membership and non-membership functions of the IF goal are linear, a solution with respect to the degree of attainment of the aggremented IF goal can be obtained by solving the following certainty mathematical problem

P3 : min γ

subject to the constraints

$$\sum_{j=1}^{n} \frac{a_{ij}}{\bar{a} - \underline{a}} y_j - \frac{\underline{a}}{\bar{a} - \underline{a}} \le \gamma; \ i = 1,2,\ldots,m \qquad (19)$$
$$e^T y = 1; \quad y \ge 0$$

where γ denotes the maximum acceptance degree of constraints fixed by the player *II*. Similarly,

P4: : max δ

subject to the constraints

$$\sum_{j=1}^{n} \frac{a_{ij}}{\underline{a} - \bar{a}} y_j - \frac{\bar{a}}{\underline{a} - \underline{a}} \ge \delta; \ i = 1,2,\ldots,m \qquad (20)$$
$$\gamma \le \delta; \ \gamma + \delta \le 1; \ \gamma, \delta \ge 0$$

where δ denotes the minimal rejection degree of constraints fixed by the player *II*. These two models (19) and (20) can be changed into the following single non-fuzzy optimization model as

(21)

$$\left.\begin{array}{l}
\max\ (\delta-\gamma-\lambda_2) \\
\sum_{j=1}^{n}\dfrac{a_{ij}}{\bar{a}-\underline{a}}y_j-\dfrac{\underline{a}}{\bar{a}-\underline{a}}\le\gamma;\ \ i=1,2,\dots,m \\
\sum_{j=1}^{n}\dfrac{a_{ij}}{\bar{a}-\underline{a}}y_j+\dfrac{\bar{a}}{\bar{a}-\underline{a}}\ge\delta;\ \ i=1,2,\dots,m \\
\left[\sum_{j=1}^{n}a_{ij}y_j+c_2\right]\ge\lambda_2;\ \ i=1,2,\dots,m \\
\left[\sum_{j=1}^{n}a_{ij}y_j+c_2\right]\le\lambda_2^{'};\ \ i=1,2,\dots,m \\
\gamma\le\delta;\ \ \gamma+\delta\le 1;\ \ \gamma,\delta\ge 0 \\
e^T y=1;\quad y\ge 0.
\end{array}\right\}$$

Therefore in contrast with what has been mentioned, the degrees of attainment of two players can not be equal. Thus, once an optimal solutions (x^*,α^*,β^*) and (y^*,γ^*,δ^*) of the mathematical programming problems (as in Theorems 2 and 3) has been obtained, (x^*,α^*,β^*) and (y^*,γ^*,δ^*) gives an equilibrium solution of the matrix game. The degree of attainment of IF goals G_1 and G_2 can then be determined by evaluating $x^{*T}Ay^*$ and $x^{*T}By^*$, then employing the membership and non-membership function μ_1,γ_1 and μ_2,γ_2.

Note: Now, if (x^*,α^*,β^*) and (y^*,γ^*,δ^*) are optimal solutions of (18) and (21) respectively, then for $\alpha^*=\beta^*=\gamma^*=\delta^*$, the IF matrix game reduce to the crisp matrix game *G*. In this case, the constraints of (18) and (21) become

$$\sum_{i=1}^{m}a_{ij}x_i^*\ge\bar{a};\ \ j=1,2,\dots,\ n$$

$$\sum_{i=1}^{m}x_i^*=1;\ \ x^*\ge 0$$

and

$$\sum_{j=1}^{n}a_{ij}y_j^*\le\underline{a};\ i=1,2,\dots,m$$

$$\sum_{j=1}^{n}y_j^*=1;\ \ y^*\ge 0,$$

which do not guarantee the fact that x^* and y^* are optimal strategies for the player *I* and player *II* respectively. From the above constraints, we see that, this can happen only when $\underline{a}=\bar{a}$ = the value of the game. In this case, (18) and (21) are not defined and $\underline{a}-\bar{a}=0$. Also, this relation is not possible, as

$$\bar{a}=\max_i\max_j a_{ij};\ \underline{a}=\min_i\min_j a_{ij} \tag{22}$$

so that A is a constant matrix. Thus the construction of the matrix game with IF goal as presented here seems to the more generalization than Nishizaki and Sakawa [5].

IV. NUMERICAL EXAMPLE

Consider the two person zero-sum crisp matrix game *G*, whose pay-off matrix *A* is $A=\begin{pmatrix}1&3\\4&0\end{pmatrix}$.The solution of the game *G* is $x^*=\left(\frac{2}{3},\frac{1}{3}\right)^T$, $y^*=\left(\frac{1}{2},\frac{1}{2}\right)^T$ and the value of the game is v = 2. Next we consider the IF versions of the game G. The results using equations (18) and (21) are given in the following tabular form :

α	β	x_1	x_2	γ	δ	y_1	y_2
0.10	0.09	0.13	0.87	0.10	0.70	0.90	0.10
0.20	0.18	0.27	0.73	0.20	0.65	0.80	0.20
0.30	0.28	0.40	0.60	0.30	0.60	0.70	0.30
0.40	0.37	0.53	0.47	0.40	0.55	0.60	0.40
0.50	0.50	0.67	0.33	0.50	0.50	0.50	0.50

In general, the solution of the IFO problem is different from the solution of the analogous fuzzy problems, and the degrees of satisfaction of the given objective or constraint in an IFO problem can be higher or lower. This depends on the formulation of the respective functions of acceptance and rejection.

(i) The optimal solutions, obtained with formulating the problem by the help of equations (18) and (21), for $\alpha = \beta = \gamma = \delta = 0.5$ are $x^* = \left(\frac{2}{3}, \frac{1}{3}\right)^T$, $y^* = \left(\frac{1}{2}, \frac{1}{2}\right)^T$. In this case, both the players have the same aspiration lavels, i.e. $\lambda_1 = \lambda_2$ and therefore the fuzzy game coincides with the crisp two person zero sum game *G*.

(ii) Let $\alpha = \gamma =$ 0.1, in this case, the optimal solutions of these LPP problems are $x^* = (0.13, 0.87)^T$, $y^* = (0.90, 0.10)^T$. This choice corresponds to the situation where player *I* aspires to win more than 3.60, but is satisfied, if he/she wins more than 2.61. Similarly, player *II* aspires not to loose more than 3.60 but he/she will be satisfied if he/she loses at most 1.10.

V. CONCLUSION

In this paper, we have presented a model for studying two person matrix games with IF goals. In this approach, the degree of acceptance and the degree of rejection of objective and constraints respectively are introduced together, where one cannot be simply considered as a complement of the other and the sum of their value is less than or equal to 1. Since the strategy spaces of player *I* and player *II* could be polyhedral sets, we may also conceptualize constrained IF matrix game on the lines of crisp constrained matrix games. On such a basis, we have defined the solution in terms of degree of attainment of IF goal in IFS environment, and found it by solving a pair of LPP. A numerical example has illustrated the proposed methods. This theory can be applied in decision making procedures in areas such as economics, operation research, management, war science, etc.

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