International Journal of Engineering Research and Development
e-ISSN: 2278-067X, **p**-ISSN : 2278-800X, www.ijerd.com
Volume 4, Issue 12 (November 2012), PP. 30-36

Application of Matrix Iteration for Determining the Fundamental Frequency of Vibration of a Continuous Beam

S. Sule[1], T.C. Nwofor[2]
Department of Civil and Environmental Engineering, University Of Port Harcourt,
P.M.B. 5323, Rivers State, Nigeria.

Abstract:-In the previous paper, the use of matrix iteration to determine the natural frequencies of vibration of continuous beam system using the concept of wave propagation in a prismatic bar is reported. The natural frequencies of vibration obtained from the formulated model were compared with the values obtained from literature and those of the exact solution. An error of 28% and 30% was observed when the fundamental frequency of vibration obtained from the previous model was compared with that obtained from literature and the exact solution. In the present study, an improved mathematical model is formulated to determine the fundamental frequency of vibration of a continuous beam as a structural system with distributed mass using mode shape concept. In the course of the system's oscillation, the displacements produced by the force of inertia is assumed to have the shape of a particular vibration mode and in harmonic with the particular modal frequency. A numerical example was given to demonstrate the applicability of the present model. The fundamental frequency value obtained from the present model was found to improve by 27.4% and almost identical to the exact fundamental frequency value and that obtained literature.

Keywords: Mathematical model, natural frequencies, fundamental frequency, vibration mode, inertia

I. INTRODUCTION

Resonance is a very dangerous phenomenon that occurs if the fundamental value of the natural frequencies of vibration is exceeded by the frequency of excitation. Resonance results in large amplitude displacement of structures leading to development of large stresses and strains in the affected structure that may eventually result to structural failure thereby affecting the performance of the structure or structures in service. The dynamic analysis of a continuous beam as a system with distributed mass to get the fundamental vibration frequency is always cumbersome because of the difficulty involved in the mathematical manipulations. The difficulty that is encountered in the mathematical process is due to infinite number of degrees of freedom [1-9]. The dynamic analysis is made simpler using lumped mass concept. The original beam with distributed mass is now converted to a weightless system with masses lumped at chosen points called the nodal points. The weightless beam system now has a finite number of degrees of freedom [10]. The degree of freedom is numerically equal to the number of independent geometric parameters that describes the positions of all masses for all possible displacements of the structural system at any point in time. The present model is said to be defined if the nodal lumped masses and their coordinates are known [11].

In this paper, matrix iteration is employed to determine the fundamental frequency of vibration of a continuous beam system undergoing self excited vibration. The algorithm involved is simple and can be achieved manually most especially when finite number of degrees of freedom is involved.

II. 2. **Formulation of Mathematical Model**

For an undamped MDOF beam system (Figure 1) with n degrees of freedom, displacements are assumed to be linear functions of the forces of inertia, the equations of motion of the lumped masses at chosen nodal points are given by:

$$\begin{aligned}
-X_1 &= m_1\ddot{X}_1\delta_{11} + m_2\ddot{X}_2\delta_{12} + \dots . m_n\ddot{X}_n\delta_{1n} \\
-X_2 &= m_2\ddot{X}_2\delta_{21} + m_2\ddot{X}_2\delta_{22} + \dots . m_n\ddot{X}_n\delta_{2n} \\
- \quad & - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\
- \quad & - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\
-X_n &= m_n\ddot{X}_n\delta_{n1} + m_2\ddot{X}_2\delta_{n2} + \dots . m_n\ddot{X}_n\delta_{nn}
\end{aligned} \tag{1}$$

where:

$m\ddot{X}_i$ = Inertia force generated by an ith particular oscillating mass.

X_i = displacement produced at ith mode by inertia force generated by an ith oscillating mass.

δ_{ij}, δ_{ii} = direct and indirect flexibility coefficient respectively.

The flexibility coefficients resulting from the forces of inertia at the individual nodal points are given by:

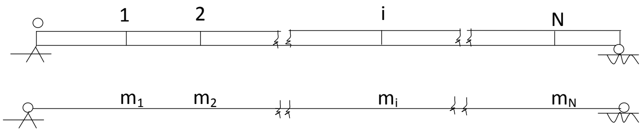


Figure 1: Lumped masses at beam nodes.

$$\delta_{ij} = \sum_{i=1}^{n} \frac{1}{EI} \int_o^L m_i m_j dx \tag{2}$$

The equation for undamped natural vibration frequency is given by :

$$K = \omega^2 m \tag{3}$$

For an ith vibration mode equation (3) transforms to:

$$K = \omega_i^2 m \tag{4}$$

Multiplying both sides of equation (4) by an ith displacement X_i caused by an ith force of inertia X_i transforms equation (4) to:

$$KX_i = \omega_i^2 X_i m \tag{5}$$

The displacement that is produced by an ith oscillating mass has the configuration of an ith vibration mode and also in harmonic with an ith modal frequency. Therefore,

$$KX_i = F_i \tag{6}$$

Equation (5) now becomes:

$$F_i = \omega_i^2 X_i m \tag{7}$$

From equation (6),

$$X_i = \frac{F_i}{K} = F_i K^{-1} \tag{8}$$

where:

$F_i =$ Force of inertia at ith mode

K^{-1} Inverse of stiffness matrix

Substituting for F_i in equation (8) using equation (7) gives:

$$X_i = \omega_i^2 m K^{-1} X_i \tag{9}$$

Let the row vector of n dimensions

$$X_i = [X_1, X_2, \ldots, X_n]^T \tag{10}$$

represent the vector of displacement in the ith vibration mode.
From structural mechanics,

$$K^{-1} = \delta \tag{11}$$

where:

$\delta =$ flexibility coefficient matrix.

Without loss of generality,

$$K_{ii}^{-1} = \delta_{ii} \tag{12}$$

and

$$K_{ij}^{-1} = \delta_{ij} \tag{13}$$

Equation (7) now transforms to:

$$X_i = \omega_i^2 m \delta X_i \tag{14}$$

Let

$$m\delta = h \tag{15}$$

Equation (13) represents the dynamic charcateristics of the weightless beam subjected to oscillating masses.
Therefore, from equation (12),

$$X_i = \omega_i^2 h X_i \tag{16}$$

For the first vibration mode i=1, and equation (5) becomes:

$$X_1 = \omega_1^2 h X_1^o \tag{17}$$

where:

$$X_1^o = (1 \quad 1 \quad 1)^T \tag{18}$$

represents an arbitrarily chosen initial displacement vector.

Let y_o represent non-zero nth order displacement vector and let max $y_{o\,\max}$ represent its largest displacement element. Let X_o be the vector obtained by when the entries of y_o are scaled by $y_{o\,\max}$.

Using equation (16),

$$\Rightarrow X_o = \frac{y_o}{y_{o\max}} \tag{19}$$

The generalized sequence of improved displacement vectors are given by equations (18) and (19).

$$X_i = \frac{y_k}{y_{k\max}};\ k = 0, 1, \ldots, n \tag{20}$$

$$hX_k = h^T X_k = [h]\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = y_{k+1} \tag{21}$$

Let $\{y_k\}_1^\infty$ be a sequence of approximations to y_n with

$$\lim_{k \Rightarrow \infty} (y_k) = \lambda_n \tag{22}$$

where:

$$\lambda_n = \omega_n^2 (n = 1, 2, \ldots) \tag{23}$$

From equations (19) and (21), the natural frequency corresponding to a given vibration mode is given by:

$$\omega_n = (y_{k+1})^{0.5} \tag{24}$$

and the fundamental value (n=1) is given by

$$\omega_1 = (y_{k+1})^{0.5} \tag{25}$$

From statical consideration, the ith modal mass at ith nodal point over an ith weightless beam segment is given by:

$$m_i = \frac{\rho}{2}(l_i + l_j) \tag{26}$$

where:

ρ = distributed mass intensity of the beam in Kg/m.

III. CONCLUSION AND RESULTS

An Example for Numerical Study

A numerical example is used to demonstrate the applicability of the present formulation. A simple supported uniform beam having a distributed mass intensity of 4.75Kg/m as shown in Figure 2 is used for this numerical study[1].

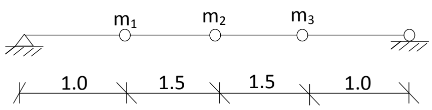


Figure 2: A 3 degrees of freedom beam system for numerical study.

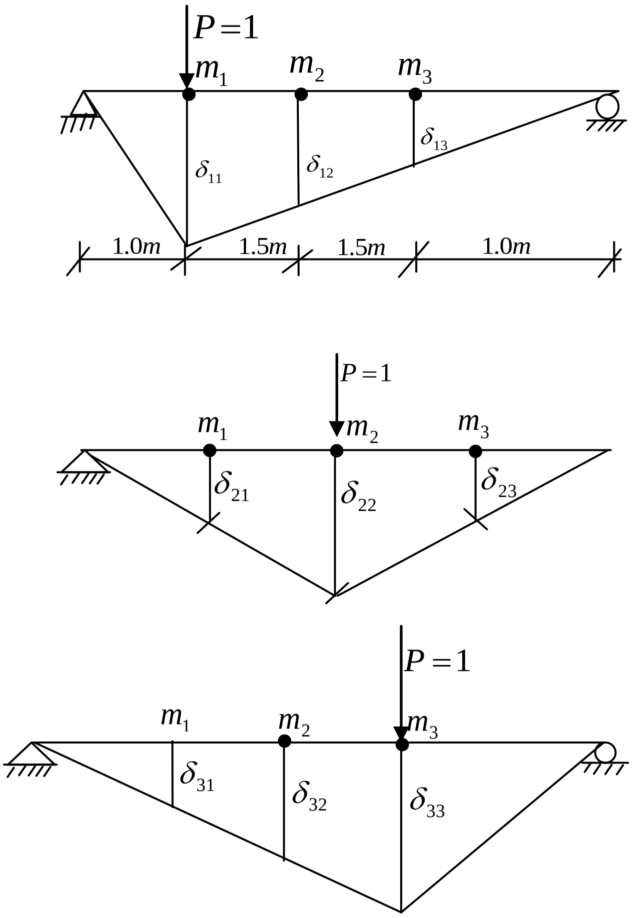


Figure 2: Derivation of flexibility factors

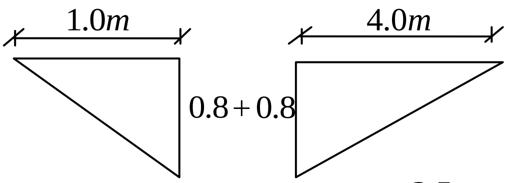
The flexibility matrix is symmetric. Therefore,

$$\delta_{ij} = \delta_{ji}$$

Using equation (2), the flexibility factors are obtained as follows:

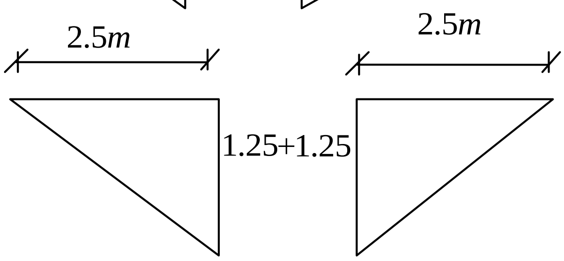
$$EI\delta_{11} = \int_0^l m_1^2 dx =$$

$$\delta_{11} = \frac{1.070}{EI}$$

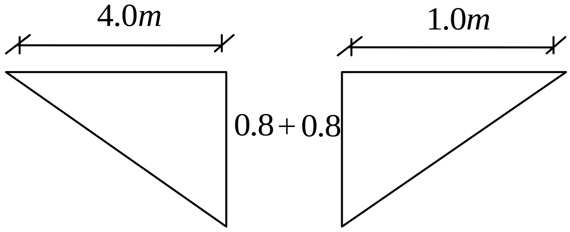


$$E\delta_{22} = \int_0^l m_2^2 dx =$$

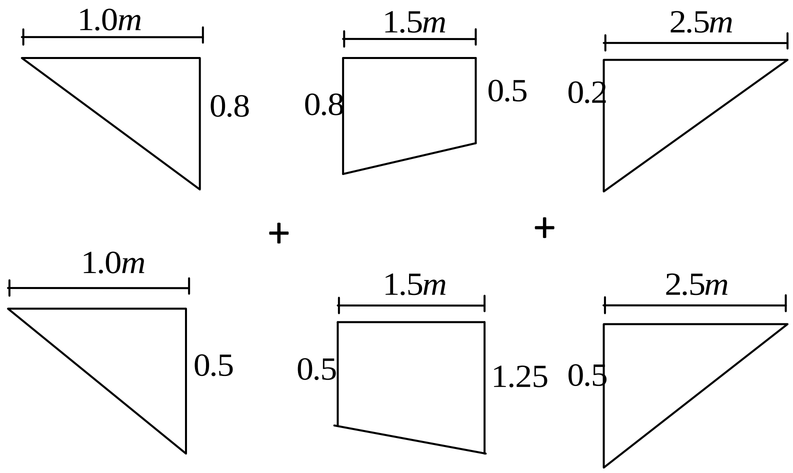
$$\delta_{22} = \frac{2.604}{EI}$$



$$\delta_{33} = \frac{1}{Ei}\int_o^l m_3^2 dx =$$

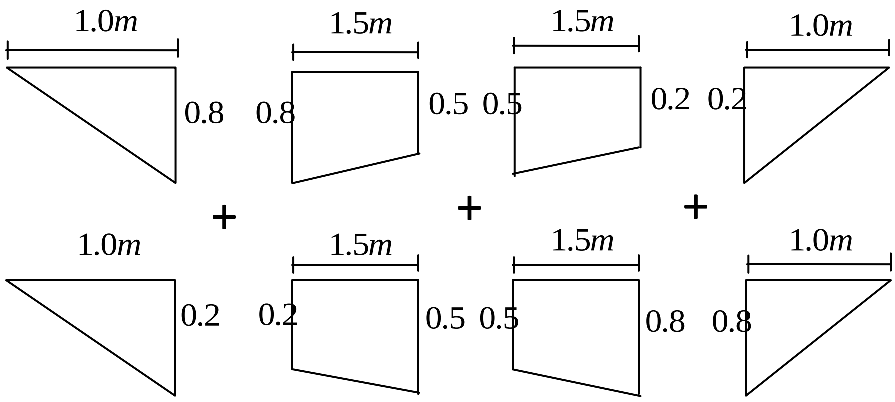


$$\delta_{12} = \delta_{21} = \frac{1}{EI}\int_o^l m_1 m_2 dx =$$



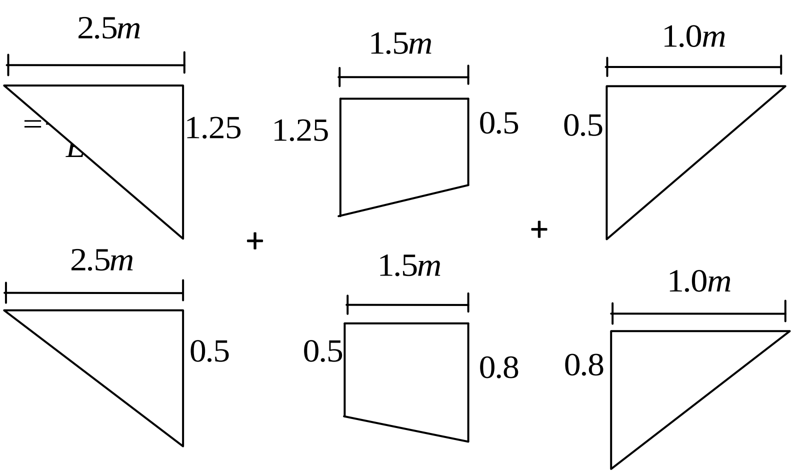
$$= \frac{1.480}{EI}$$

$$\delta_{13} = \delta_{31} = \frac{1}{EI}\int_o^l m_1 m_3 =$$



$$= \frac{0.707}{EI}$$

$$\delta_{23} = \delta_{32} = \frac{1}{EI}\int_o^l m_2 m_3 dx =$$



From equation (26),

$$m_1 = m_3 = \frac{\rho + 1.5\rho}{2} = 1.25\rho$$

$$m_2 = \frac{1.5\rho + 1.5\rho}{2} = 1.5\rho$$

The flexibility coefficients at the three nodal points are now arranged in matrix form as follows:

$$\begin{bmatrix} 1.070 & 1.480 & 0.770 \\ 1.480 & 2.604 & 1.480 \\ 0.770 & 1.480 & 1.070 \end{bmatrix}$$

From equation (15), the dynamic matrix is given by:

$$h = \frac{1}{EI}\begin{bmatrix} 1.070 & 1.480 & 0.770 \\ 1.480 & 2.604 & 1.480 \\ 0.770 & 1.480 & 1.070 \end{bmatrix}\begin{bmatrix} 1.25\rho & 0 & 0 \\ 0 & 15\rho & 0 \\ 0 & 0 & 1.25\rho \end{bmatrix}$$

$$\Rightarrow h = \frac{1}{EI}\begin{bmatrix} 1.34\rho & 2.22\rho & 0.96\rho \\ 1.85\rho & 3.91\rho & 1.85\rho \\ 0.96\rho & 2.22\rho & 1.34\rho \end{bmatrix}$$

Using equation (18), multiplication of dynamic matrix with assumed initial unit displacement vector gives:

$$X_1 = \frac{1}{EI}\begin{bmatrix} 1.34\rho & 2.22\rho & 0.96\rho \\ 1.85\rho & 3.91\rho & 1.85\rho \\ 0.96\rho & 2.22\rho & 1.34\rho \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 4.52\rho \\ 7.61\rho \\ 4.52\rho \end{bmatrix}$$

Using 7.61ρ as the maximum displacement in the non zero displacement vector, the improved displacement vector is given by:

$$X_2 = \frac{1}{EI}(3.59\rho, 6.10\rho, 3.59\rho)^T$$

Again, the maximum displacement is 6.10ρ . The new improved displacement is:

$$X_3 = \frac{1}{EI}(3.575\rho, 6.09\rho, 3.575\rho)^T$$

Using equation (23),

$$\omega_1 = \sqrt{\frac{EI}{6.09\rho}} = 0.405\sqrt{\frac{EI}{\rho}}$$

Table 1: Comparison of results

Fundamental frequency	**Present model**	**Sule 2009**	**Osadebe 1999**	**Exact solution**
	$0.4050\sqrt{\frac{EI}{\rho}}$	$0.515\sqrt{\frac{EI}{\rho}}$	$0.4039\sqrt{\frac{EI}{\rho}}$	$0.3948\sqrt{\frac{EI}{\rho}}$

IV. DISCUSSION OF RESULTS

Table 1 shows the comparison of result obtained from the present model with Sule [2], Osadebe [1] and the exact solution [3]. The percentage errors of 27.5% and 30.45% in the previous model[2] compared with the control points [1] and exact solution [4] have reduced to 0.272% and 0.258% in the present formulation showing the effectiveness of the present model in the determination of the fundamental frequency of vibration of a continuous beam. The disparity between the result of the present model and the previous model[1], Osadebe [2] and the exact solution[3] may be due to difference in the assumptions used in the model formulation.

V. CONCLUSION

In conclusion, the present model produces a result that is almost identical with those of the control points [1] and [4] and improves on the previous result of fundamental frequency [2] by 27.4%, showing higher predictive ability of the present model. The present model can be used to predict the fundamental frequency of vibration of a multi-storey building.

REFERENCES

[1]. Osadebe N.N. (1999). An improved MDOF model simulating some system with distributed mass. Journal of University of Science and Technology, Kumasi, volume 19, No. 1, 2 and 3.

[2]. Sule, S. (2011). Approximate method for the determination of natural frequencies of a multi-degree of freedom beam system. Nigerian Journal of Technology, vol.30, No.2, June 2011, pp. 43-49.

[3]. Rao, S.S. (2003). Mechanical vibrations, 4th edition, Prentice Hall, Inc.

[4]. Thomson, W.T. (1988). Theory of Vibrations with applications. 3rd edition, CBS Publishers New Delhi.

[5]. Humar, J.L. (1990). Dynamics of Structures, Prentice Hall, Inc.

[6]. De Silva, C.W. (1999). Vibration Fundamentals and Practice, CRC Press.

[7]. Chopra, A.K. (2001). Dynamics of Structures, theory and applications to earthquake engineering, 2nd edition, Prentice Hall Inc.

[8]. Clough R.W. and Penzien, J. (1982). Dynamics of Structures, McGraw-Hill Int. Student Edition, Tomyo.

[9]. Smith, B.S. and Coull, A. (1999). Tall building structures: Analysis and Design, Wiley, New York.

[10]. Briggs, I.M. (1964). Introduction to structural dynamics McGraw-Hill Book Company, New York.

[11]. Den Hartog, J.P. (1956). Mechanical vibrations, McGraw Hill Book Company, New York, pp. 122-169.