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On Generalized Polynomials I

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Abstract:-Voronowskajain 1932 proved his result forBernstein polynomial. We have extended the corresponding result of Voronowskaja for Lebesgueintegrable function in L_1-norm by our newly defined Generalized Polynomial.

Keywords:-Bernstein Polynomials, Convergence, Generalized Polynomials, Integrable functions, $\boldsymbol{L_1}$-norm.

I. INTRODUCTIONAND RESULTS

If $f(x)$ is a function defined on [0,1], the Bernstein polynomial $B_n^f(x)$ of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x) \quad (1.1)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (1.2)$$

One question arises about the rapidity of convergence of B_n^f(x)to $f(x)$. An answer to this question has been given in different directons. One direction is that in which $f(x)$ is supposed to be at least twice differentiable in a point x of [0,1]. Voronowskaja [6] proved that

$$\lim_{n\to\infty} n \left| f(x) - B_n^f(x) \right| = -\frac{1}{2} x(1-x) f''(x) \quad (1.3)$$

In particular,if $f''(x) \neq 0$, difference $f(x) - B_n^f(x)$ is exactly of order n^{-1}

A small modification of Bernstein polynomial due to Kantorovitch [4] makes it possible to approximate Lebesgueintegrable function in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt \right\} p_{n,k}(x) \quad (1.4)$$

where$p_{n,k}(x)$ is defined by (1.2)

By Abel’s formula(see Jensen [3])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^{n} \binom{n}{k}\, x(x+k\alpha)^{k-1} y(y+(n-k)\alpha)^{n-k-1} \quad (1.5)$$

If we put $y = 1-x$, we obtain (see Cheney and Sharma [2])

$$1 = \sum_{k=0}^{n} \binom{n}{k} \frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \quad (1.6)$$

Thus defining

$$q_{n,k}(x;\alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \quad (1.7)$$

we have

$$\sum_{k=0}^{n} q_{n,k}(x;\alpha) = 1 \qquad (1.8)$$

we now define a polynomial named as Generalized Polynomial

$$U_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha) \quad (1.9)$$

where $q_{n,k}(x;\alpha)$ is defined in (1.7) and moreover when $\alpha = 0$, (1.7) and (1.9) reduces to (1.2) and (1.4) respectively.

The function

$$S(\mathrm{v},n,x,y) = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k+\mathrm{v}-1} y(y+(n-k)\alpha)^{n-k-1} \quad (1.10)$$

satisfies the reduction formula

$$S(\mathrm{v},n,x,y) = xS(\mathrm{v}-1,n,x,y) + n\alpha S(\mathrm{v},n-1,x+\alpha,y) \quad (1.11)$$

from (1.5) & (1.10) we can have

$$xS(0,n,x,y) = (x+y)(x+y+n\alpha)^{n-1} \ ,$$

by repeated use of reduction formula(1.11) and (1.5) we get

$$S(1,n,x,y) = \sum_{k=0}^{n}\binom{n}{k} k!\ \alpha^{n}(x+y)(x+y+n\alpha)^{n-k-1}, \quad (1.12)$$

$$S(2,n,x,y) = \sum_{k=0}^{n}\binom{n}{k} (x+k\alpha)k!\ \alpha^{n} S(1,n-k,x+k\alpha,y). \quad (1.13)$$

Since $k! = \int_0^{\infty} e^{-t}t^{k}dt$ and so using binomial expansion we obtain

$$S(1,n,x,y) = \int_0^{\infty} e^{-t}(x+y)(x+y+n\alpha+t\alpha)^{n-1}dt, \quad (1.14)$$

$$S(2,n,x,y) = \int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[\,(x+y)(x+y+n\alpha+t\alpha+s\alpha)^{n-1}dt \quad n\alpha^{n} S\,(x+y+n\alpha+t\alpha+s\alpha)^{n-2}], \quad (1.15)$$

we, therefore , can show

$$S(1,n-1,x+\alpha,1-x) = \int_0^{\infty} e^{-t}(1+n\alpha+t\alpha)^{n-1}dt\ , \quad (1.16)$$

$$S(1,n-2,x+\alpha,1-x+\alpha) = \int_0^{\infty} e^{-t}\alpha(1+n\alpha+t\alpha)^{n-2}dt, \quad (1.17)$$

$$S(2,n-2,x+2\alpha,1-x) = \int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[\,(x+2\alpha)(1+n\alpha+t\alpha+s\alpha)^{n-2}dt + (n-2)\alpha^{2} S\,(1+n\alpha+ t\alpha+s\alpha n-3], \quad (1.18)$$

$$S(2,n-3,x+2\alpha,1-x+\alpha) = \int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}[\,(x+2\alpha)(1+n\alpha+t\alpha+s\alpha)^{n-3}ds + (n-3)\alpha^{2} S(1+n\alpha+t\alpha+s\alpha)^{n-4}]. \quad (1.19)$$

In this paper, we shall prove the corresponding result of approximation due to Voronowskja [6] for Lebesgueintegrable function in L_1-norm by ourGeneralized polynomial (1.9). In fact we state our result as follows:

Theorem:let $f(x)$ be bounded Lebesgueintegrable function with its first derivative in [0,1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0,1], then for $\alpha = \alpha_n = o(1/n)$,

$$\lim_{n\to\infty} n\left[U_n^{\alpha}(f,x) - f(x)\right] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

II. LEMMAS

In order to prove our result we need the following lemmas

Lemma 2.1: – For all value of x

$$\sum_{k=0}^{n} kq_{n,k}(x;\alpha) \leq \frac{1+n\alpha}{1+\alpha}nx - \frac{n(n-1)x\alpha}{1+2\alpha}$$

Proof:

$$\sum_{k=0}^{n} kq_{n,k}(x;\alpha) = \sum_{k=0}^{n} k\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

$$= nx\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

$$= nx\sum_{\mu=0}^{n-1}\binom{n-1}{\mu}\frac{(x+\mu\alpha+\alpha)^{\mu}(1-x)(1-x+(n-\mu-1)\alpha)^{n-\mu-2}}{(1+n\alpha)^{n-1}}$$

$$= \frac{nx}{(1+n\alpha)^{n-1}}\{\sum_{\mu=0}^{n-1}\binom{n-1}{\mu}(x+\mu\alpha+\alpha)^{\mu}(1-x+(n-\mu-1)\alpha)^{n-\mu-2}$$

$$-(n-1)\alpha\sum_{\mu=0}^{n-2}\binom{n-2}{\mu}(x+\mu\alpha+\alpha)^{\mu}(1-x+(n-\mu-2)\alpha)^{n-\mu-2}\}$$

$$=\frac{nx}{(1+n\alpha)^{n-1}}[s(1,n-1.x+\alpha,1-x)-(n-1)\alpha s(1,n-2,x+\alpha,1-x+\alpha]\text{by (1.14)}$$

$$=\frac{nx}{(1+n\alpha)^{n-1}}[\int_0^{\infty}e^{-t}(1+n\alpha+t\alpha)^{n-1}dt-(n-1)\alpha\int_0^{\infty}e^{-t}(1+n\alpha+t\alpha)^{n-2}dt] \qquad \text{by (1.16)\&(1.17)}$$

$$=\frac{nx}{(1+n\alpha)^{n-1}}\int_0^{\infty}e^{-t}(1+n\alpha+t\alpha)^{n-1}dt-\frac{n(n-1)x\alpha}{(1+n\alpha)^{n-1}}\int_0^{\infty}e^{-t}(1+n\alpha+t\alpha)^{n-2}dt$$

$$=\frac{nx}{(1+n\alpha)^{n-1}}\int_0^{\infty}e^{-t}\left(1+\frac{t\alpha}{1+n\alpha}\right)^{n-1}(1+n\alpha)^{n-1}dt$$

$$-\frac{n(n-1)x\alpha}{(1+n\alpha)^{n-1}}\int_0^{\infty}e^{-t}\left(1+\frac{t\alpha}{1+n\alpha}\right)^{n-2}(1+n\alpha)^{n-2}dt$$

$$=nx\int_0^{\infty}e^{-t}\left(1+\frac{t\alpha}{1+n\alpha}\right)^{n-1}dt -- \frac{n(n-1)x\alpha}{1+n\alpha}\int_0^{\infty}e^{-t}\left(1+\frac{t\alpha}{1+n\alpha}\right)^{n-2}dt$$

$$= nx\int_0^{\infty}e^{-\frac{1+n\alpha}{\alpha}u}(1+u)^{n-1}\frac{1+n\alpha}{\alpha}du -$$

$$\frac{n(n-1)x\alpha}{1+n\alpha}\int_0^{\infty}e^{-\frac{1+n\alpha}{\alpha}u}(1+u)^{n-2}\frac{1+n\alpha}{\alpha}du$$

$$=\frac{(1+n\alpha)nx}{\alpha}\int_0^{\infty}e^{-(\frac{1}{\alpha}+n)u}(1+u)^{n-1}du-\frac{n(n-1)x\alpha}{\alpha}\int_0^{\infty}e^{-(\frac{1}{\alpha}+n)u}(1+u)^{n-2}du$$

$$\leq\frac{(1+n\alpha)nx}{\alpha}\int_0^{\infty}e^{-(\frac{1}{\alpha}+n)u}e^{(n-1)u}du-\frac{n(n-1)x\alpha}{\alpha}\int_0^{\infty}e^{-(\frac{1}{\alpha}+n)u}e^{(n-2)u}du$$

$$=\frac{(1+n\alpha)nx}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+1)u}du-\frac{n(n-1)x\alpha}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+2)u}du$$

$$=\frac{(1+n\alpha)nx}{1+\alpha}\int_0^{\infty} e^{-v}dv-\frac{n(n-1)x\alpha}{1+2\alpha}\int_0^{\infty} e^{-w}dw$$

$$=\frac{(1+n\alpha)nx}{1+\alpha}-\frac{n(n-1)x\alpha}{1+2\alpha}$$

which completes the proof of the lemma 2.1.

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n} k(k-1)q_{n,k}(x;\alpha)\leq n(n-1)[(x+2\alpha)\{\frac{1+n\alpha}{(1+2\alpha)^2}-\frac{(n-2)\alpha}{(1+3\alpha)^2}\}$$

$$+(n-2)\alpha^2\{\frac{1+n\alpha}{(1+3\alpha)^3}-\frac{(n-3)\alpha}{(1+4\alpha)^3}\}]$$

Proof:

$$\sum_{k=0}^{n} k(k-1)q_{n,k}(x;\alpha)\leq n(n-1)x\sum_{k=1}^{n}\binom{n-2}{k-2}\frac{(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}\sum_{k=1}^{n}\binom{n-2}{v}(x+k\alpha+2\alpha)^{v+1}(1-x)(1-x+(n-v-2)\alpha)^{n-k-3}$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}[S(2,n\text{-}2,x+2\alpha,1\text{-}x)\text{-}(n\text{-}2)\alpha\, S(2,n\text{-}3,x+2\alpha,1\text{-}x+\alpha)]$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}[S(2,n\text{-}2,x+2\alpha,1\text{-}x)]-\frac{n(n-1)(n-2)x\alpha}{(1+n\alpha)^{n-1}}[S(2,n\text{-}3,x+2\alpha,1\text{-}x+\alpha)]$$

$$=I_1-I_2 \qquad (2.2.1)$$

$$I_1=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}[S(2,n\text{-}2,x+2\alpha,1\text{-}x)]$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[(x+2\alpha)(1+n\alpha+s\alpha+t\alpha)^{n-2}$$

$$+(n-2)\alpha^2 s(1+n\alpha+s\alpha+t\alpha)^{n-3}]$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[(x+2\alpha)(1+n\alpha+s\alpha+t\alpha)^{n-2}]$$

$$+\frac{n(n-1)(n-2)x\alpha^2}{(1+n\alpha)^{n-1}}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[s(1+n\alpha+s\alpha+t\alpha)^{n-3}]$$

$$=I_{1.1}+I_{1.2}(\ 2.2.2)$$

$$I_{1.1}=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[(x+2\alpha)(1+n\alpha+s\alpha+t\alpha)^{n-2}]$$

$$=\frac{n(n-1)x}{(1+n\alpha)^{n-1}}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+n\alpha}\right)^{n-2}(1+n\alpha)^{n-2}]$$

$$=\frac{n(n-1)x}{1+n\alpha}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}ds[(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+n\alpha}\right)^{n-2}]$$

$$\leq\frac{n(n-1)x(x+2\alpha)}{1+n\alpha}\int_0^{\infty} e^{-t}dt\int_0^{\infty} e^{-s}e^{(n-2)(\frac{s\alpha+t\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)x(x+2\alpha)}{1+n\alpha}\int_0^\infty e^{-t+(\frac{(n-2)t\alpha}{1+n\alpha})}dt\int_0^\infty e^{-s+(\frac{(n-2)s\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)x(x+2\alpha)}{1+n\alpha}\int_0^\infty e^{-t(\frac{1+2\alpha}{1+n\alpha})}dt\int_0^\infty e^{-s(\frac{1+2\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)x(x+2\alpha)}{1+n\alpha}\int_0^\infty e^{-u}du\,\frac{1+n\alpha}{1+2\alpha}\int_0^\infty e^{-v}dv\,\frac{1+n\alpha}{1+2\alpha}$$

$$=\frac{n(n-1)x(x+2\alpha)}{(1+2\alpha)^2}(1+n\alpha)\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$$

$$=\frac{n(n-1)x(x+2\alpha)}{(1+2\alpha)^2}(1+n\alpha)\text{(2.2.3)}$$

$$I_{1.2}=\frac{n(n-1)(n-2)x\alpha^2}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[s(1+n\alpha+s\alpha+t\alpha)^{n-3}]$$

$$=\frac{n(n-1)(n-2)x\alpha^2}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}\left(1+\frac{s\alpha+t\alpha}{1+n\alpha}\right)^{n-3}(1+n\alpha)^{n-3}ds$$

$$\leq\frac{n(n-1)(n-2)x\alpha^2}{(1+n\alpha)^2}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n-3)(\frac{s\alpha+t\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)x\alpha^2}{(1+n\alpha)^2}\int_0^\infty e^{-t(\frac{1+3\alpha}{1+n\alpha})}dt\int_0^\infty se^{-s(\frac{1+3\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)x\alpha^2)}{(1+3\alpha)^3}(1+n\alpha)\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$$

$$=\frac{n(n-1)(n-2)x\alpha^2)}{(1+3\alpha)^3}(1+n\alpha)\text{(2.2.4)}$$

from (2.2.2) , (2.2.3) & (2.2.4) we have

$$I_1\leq(1+n\alpha)n(n-1)x\{\frac{(x+2\alpha)}{(1+2\alpha)^2}+\frac{(n-2)\alpha^2}{(1+3\alpha)^3}\qquad(2.2.5)$$

Now we evaluate

$$I_2=\frac{n(n-1)(n-2)x\alpha}{(1+n\alpha)^{n-1}}[\mathrm{S}(2,n\text{-}3,x+2\alpha,1\text{-}x+\alpha)]$$

$$=\frac{n(n-1)(n-2)x\alpha}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x+2\alpha)(1+n\alpha+s\alpha+t\alpha)^{n-3}$$

$$+(n-3)\alpha^2s(1+n\alpha+s\alpha+t\alpha)^{n-4}]\quad\text{by}\quad(1.19)$$

$$=\frac{n(n-1)(n-2)x\alpha}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,(x+2\alpha)(1+n\alpha+s\alpha+t\alpha)^{n-3}$$

$$+\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,s(1+n\alpha+s\alpha+t\alpha)^{n-4}$$

$$=I_{2.1}+I_{2.2}\text{(2.2.6)}$$

$$I_{2.1}=\frac{n(n-1)(n-2)x\alpha}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+n\alpha}\right)^{n-3}(1+n\alpha)^{n-3}$$

$$\leq\frac{n(n-1)(n-2)x(x+2\alpha)\alpha}{(1+n\alpha)^2}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}e^{(n-3)(\frac{s\alpha+t\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)x(x+2\alpha)\alpha}{(1+n\alpha)^2}\int_0^\infty e^{-t(\frac{1+3\alpha}{1+n\alpha})}dt\int_0^\infty se^{-s(\frac{1+3\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)x(x+2\alpha)\alpha}{(1+n\alpha)^2}\int_0^\infty e^{-u}du\,\frac{1+n\alpha}{1+3\alpha}\int_0^\infty e^{-v}dv\,\frac{1+n\alpha}{1+3\alpha}$$

$$=\frac{n(n-1)(n-2)x(x+2\alpha)\alpha}{(1+3\alpha)^2}\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$$

$$=\frac{n(n-1)(n-2)x(x+2\alpha)\alpha}{(1+3\alpha)^2} \qquad (2.2.7)$$

$$I_{2.2} \quad =\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}\,(1+n\alpha+s\alpha+t\alpha)^{n-4}ds$$

$$=\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^{n-1}}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}\left(1+\frac{s\alpha+t\alpha}{1+n\alpha}\right)^{n-3}ds(1+n\alpha)^{n-3}$$

$$\leq\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^3}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n-4)(\frac{s\alpha+t\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^3}\int_0^\infty e^{-t(\frac{1+4\alpha}{1+n\alpha})}dt\int_0^\infty se^{-s(\frac{1+4\alpha}{1+n\alpha})}ds$$

$$=\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+n\alpha)^3}\int_0^\infty e^{-u}du\,\frac{1+n\alpha}{1+4\alpha}\int_0^\infty v(\frac{1+n\alpha}{1+4\alpha})e^{-v}dv\,\frac{1+n\alpha}{1+4\alpha}$$

$$=\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+4\alpha)^3}\int_0^\infty e^{-u}du\int_0^\infty ve^{-v}dv$$

$$=\frac{n(n-1)(n-2)(n-3)x\alpha^3}{(1+4\alpha)^3} \qquad (2.2.8)$$

from (2.2.6), (2.2.7)&(2.2.8) , we have

$$I_2\leq n(n-1)(n-2)x\{\frac{(x+2\alpha)}{(1+3\alpha)^2}+\frac{(n-3)\alpha^3}{(1+4\alpha)^3}\}$$

thereforesubstituting the values of I_1&I_2 in (2.2.1) , we get

$$\leq(1+n\alpha)n(n-1)x\{\frac{(x+2\alpha)}{(1+2\alpha)^2}+\frac{(n-2)\alpha^2}{(1+3\alpha)^3}\}-n(n-1)(n-2)x\{\frac{(x+2\alpha)}{(1+3\alpha)^2}+\frac{(n-3)\alpha^3}{(1+4\alpha)^3}\}$$

$$=n(n-1)x[(x+2\alpha)\{\frac{(1+n\alpha)}{(1+2\alpha)^2}-\frac{(n-2)\alpha}{(1+3\alpha)^2}\}+(n-2)\alpha^2\{\frac{(1+n\alpha)}{(1+3\alpha)^2}-\frac{(n-3)\alpha}{(1+4\alpha)^3}\}$$

which completes the proof of lemma 2.2.

Lemma 2.3: - For all values of $x\in|0,1|$ and for $\alpha=\alpha_n=o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2dt\right\}q_{n,k}(x;\alpha)\leq x(1-x)/n$$

Proof :

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2dt\right\}q_{n,k}(x;\alpha$$

$$= \sum_{k=0}^{n} [x^2 - \frac{2kx + x}{n+1} + \frac{k^2 + k}{(n+1)^2} + \frac{1}{3(n+1)^2}] q_{n,k}(x;\alpha)$$

$$\leq x^2 - \frac{1}{(n+1)} \left[\frac{2(1+n\alpha)nx}{1+\alpha} - \frac{2n(n-1)x^2\alpha}{1+2\alpha} \right] - \frac{x}{n+1}$$

$$+ \frac{n(n-1)}{(n+1)^2} \ [(x+2\alpha)\{\frac{(1+n\alpha)}{(1+2\alpha)^2} - \frac{(n-2)\alpha}{(1+3\alpha)^2}\}$$

$$+ (n-2)\alpha^2 \left\{ \frac{1+n\alpha}{(1+3\alpha)^3} - \frac{n-3}{(1+4\alpha)^3} \right\} + \frac{2(1+n\alpha)nx}{(n+1)^2(1+\alpha)}$$

$$- \frac{2n(n-1)x\alpha}{(n+1)^2(1+\alpha)} + \frac{1}{3(n+1)^2} \text{by lemma 2.1 \& lemma 2.2}$$

$$= -\frac{x(1-x)}{n+1} + \frac{2(1+n\alpha)n}{(n+1)^2(1+\alpha)} x(1-x) - \frac{2n(n-1)\alpha}{(n+1)^2(1+\alpha)} x(1-x) + \frac{2(1+n\alpha)n^2 x(1-x)\alpha}{(n+1)^2(1+2\alpha)^2}$$

$$- \frac{2n^2(n-1)x(1-x)\alpha^2}{(n+1)^2(1+3\alpha)^2} + \frac{n(n-1)(n-2)\alpha^2}{(n+1)^2(1+3\alpha)^3} x(1-x) + \frac{nx^2}{n+1} - \frac{2(1+n\alpha)n^2x^2(1+3\alpha+3\alpha^2)}{(n+1)^2(1+\alpha)(1+2\alpha)^2}$$

$$- \frac{2(1+n\alpha)nx\alpha}{(n+1)^2(1+2\alpha)^2} + \frac{2n^2(n-1)x^2\alpha(1+5\alpha+7\alpha^2)}{(n+1)^2(1+2\alpha)(1+3\alpha)^2} + \frac{4n(n-1)x\alpha^2}{(n+1)^2(1+3\alpha)^2} - \frac{n(n-1)(n-2)\alpha}{(n+1)^2(1+3\alpha)^3}$$

$$-x^2(1+2\alpha) + \frac{n(n-1)(n-2)x\alpha^3}{(n+1)^2(1+3\alpha)^3} + \frac{n(n-1)(1+n\alpha)x^2\alpha}{(n+1)(1+2\alpha)^2} - \frac{n(n-1)(n-2)(n-3)x\alpha^3}{(n+1)^2(1+4\alpha)^3} + \frac{1}{3(n+1)^2}$$

$$\leq \frac{x(1-x)}{n} \text{for } \alpha = \alpha_n = o\left(\frac{1}{n}\right) \text{and for large n}$$

which completes the proof of Lemma 2.3.

III. PROOF OF THE THEOREM

Proof :

We write (in view of Taylor's Theorem)

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2[\tfrac{1}{2}f''(x) + \eta(t-x)] \qquad (3.1)$$

where $\eta(h)$ is bounded $|\eta(h)| \leq H$ for all h and converges to zero with h.

Multiplying eqn. (3.1) by $(n+1)q_{n,k}(x;\alpha)$ and integrating it from *k/(n+1)* to *(k+1)/(n+1)*, then on summing ,we get

$$(n+1)\sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt \right\} q_{n,k}(x;\alpha)$$

$$= (n+1)\sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(x)dt \right\} q_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt \right\} q_{n,k}(x;\alpha)$$

$$+\frac{1}{2}(n+1)\sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)dt \right\} q_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2\eta(t-x)dt\right\}q_{n,k}(x;\alpha)$$

$$= I_3+I_4+I_5+I_6 \text{ (say)} \qquad (3.2)$$

Now first we evaluate I_3:

$$I_3 = (n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(x)dt\right\}q_{n,k}(x;\alpha) = f(x)(3.3)$$

and then

$$I_4 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}f(t-x)f'(x)dt\right\}q_{n,k}(x;\alpha)$$

$$=\sum_{k=0}^{n}\left(\frac{2k+1}{2(n+1)}- x\right)f'(x)q_{n,k}(x;\alpha)$$

$$\leq \frac{(1-2x)}{2n}f'(x)\text{for}\alpha = \alpha_n = o(1/n)(3.4)$$

Now we evaluate I_5:

$$I_5 = \frac{1}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2f''(x)dt\right\}q_{n,k}(x;\alpha)$$

$$\leq x(1-x)f''(x)/2n \qquad \text{(by lemma 2.3)} \qquad (3.5)$$

and then in the last we evaluate I_6 :

$$I_6 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2\eta(t-x)dt\right\}q_{n,k}(x;\alpha)$$

Let $\in > 0$ be arbitrary $\delta > 0$ such that $|\eta(h)| < \epsilon$for $|h| < \delta$.
Thus breaking up the sum I_6 into two parts corresponding to those values of t for which | t - x |<δ, and those for which $|t-x| \geq \delta$ and since in the given range of t, $\left|\frac{k}{n}-x\right| \sim |t-x|$, we have

$$|I_6| \leq \in \sum_{\left|\left(\frac{k}{n}\right)-x\right|<\delta}(n+1)q_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right|$$

$$+H\sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta}(n+1)q_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\right|$$

$= I_{6.1}+I_{6.2}$ (say)

$|I_{6.1}| \leq \frac{\in}{n}\,|\{x(1-x)\}|$, for $\alpha = \alpha_n = o(1/n)$

$$I_{6.2} = (n+1)\, H \sum_{\left|\left(\frac{k}{n}\right)-x\right| \geq \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} q_{n,k}(x;\alpha)$$

$$= (n+1) \sum_{\left|\left(\frac{k}{n}\right)-x\right| \geq \delta} q_{n,k}\,(x;\alpha) \frac{1}{n+1}$$

But if $\delta = n^{-\beta}$, 0 < β < 1/2 (see also Kantorovitch [4]),
then for $\alpha = \alpha_n = o(1/n)$

$\sum_{\left|\left(\frac{k}{n}\right)-x\right| \geq n^{-\beta}} q_{n,k}\,(x;\alpha) \leq Cn^{-\nu}$ for ν>0, the constant C= C(β,ν).

whence$I_{6.2} < \frac{\epsilon}{n+1} < \epsilon/n$ for all n sufficiently large and therefore it follows

$$I_6 < \epsilon/n, \text{ for all sufficiently large n} \qquad (3.6)$$

Hence from (3.2), (3.3), (3.4), (3.5) and (3.6), we have

$$(n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt \right\} q_{n,k}(x;\alpha)$$

$$= f(x) + [\{(1-2x)f'(x) + x(1-x)f''(x)\}/2n] + (\epsilon/n)$$

and therefore, finally we get

$$\lim_{n\to\infty} n \left[U_n^{\alpha}(f,x) - f(x) \right] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

where $\epsilon \to 0 \; as \; n \to \infty$
which completes the proof of the theorem.

IV. CONCLUSIONS

The result of Voronowskaja has been extended for Lebesgueintegrable function in L_1-norm by our newly defined Generalized Polynomials $U_n^f(x)$.

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