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On The Zero-Free Regions for Polynomials

M. H. GULZAR
Department of Mathematics University of Kashmir, Srinagar 190006

Abstract: In this paper we find the zero-free regions for a class of polynomials whose coefficients or their real and imaginary parts are restricted to certain conditions.
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I. INTRODUCTION AND STATEMENT OF RESULTS

The following result, known as the Enestrom-Kakeya Theorem [5] , is well-known in the theory of distribution of zeros of polynomials:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \leq 1$.

In the literature there exist several generalizations and extensions of this result. Recently, Choo and Choi [1] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

either $a_n \geq a_{n-2} \geq ...... \geq a_3 \geq a_1$ and $a_{n-1} \geq a_{n-3} \geq ...... \geq a_2 \geq a_0$,if n is odd , or $a_n \geq a_{n-2} \geq ...... \geq a_2 \geq a_0$ and $a_{n-1} \geq a_{n-3} \geq ...... \geq a_3 \geq a_1$, if n is even. Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + a_n + |a_{n-1}| + a_{n-1} + |a_1| + a_1 - a_0} .$$

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \alpha_0 ,$$
$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0 .$$

Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + (k-1)|\alpha_n| + k\alpha_n - \alpha_0 + \beta_n - \beta_0} .$$

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \text{ j=0,1,……,n,}$$

and for some $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq ......\geq |a_1| \geq |a_0| .$$

Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{k|a_n| + (k|a_n| - |a_0|)\cos\alpha + (k|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|}.$$

M. H. Gulzar [3,4] proved the following results:

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \geq 1$, $k_2 \geq 1$, $0 < \tau_1 \leq 1$, $0 < \tau_2 \leq 1$,either $k_1 a_n \geq a_{n-2} \geq ...... \geq a_3 \geq \tau_1 a_1$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_2 \geq \tau_2 a_0$,if n is odd , or $k_1 a_n \geq a_{n-2} \geq ...... \geq a_2 \geq \tau_1 a_0$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_3 \geq \tau_2 a_1$, if n is even. Then for odd n, P(z) has all its zeros in

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_1}{|a_n|},$$

where

$$K_1 = k_1(|a_n| + a_n) + k_2(|a_{n-1}| + a_{n-1}) + 2(|a_1| + |a_0|) - (|a_n| + |a_{n-1}|) - \tau_1(a_1 + |a_1|)$$
$$- \tau_2(a_0 + |a_0|),$$

and for even n, P(z) has all its zeros in

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_2}{|a_n|},$$

where

$$K_2 = k_1(|a_n| + a_n) + k_2(|a_{n-1}| + a_{n-1}) + 2(|a_1| + |a_0|) - (|a_n| + |a_{n-1}|) - \tau_1(a_0 + |a_0|)$$
$$- \tau_2(a_1 + |a_1|).$$

Theorem F: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, j=0,1,2,……,n, such that

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0.$$

Then all the zeros of P(z) lie in

$$\left| z + (k-1\frac{\alpha_n}{a_n}) \right| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j|}{|a_n|}.$$

Theorem G: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \text{ j=0,1,……,n,}$$

and for some $k \geq 1$, $0 < \tau \leq 1$

$$k|a_n| \geq |a_{n-1}| \geq ......\geq |a_1| \geq \tau|a_0|.$$

Then P(z) has all its zeros in

$$|z + k - 1| < \frac{1}{|a_n|}\left[k|a_n|(\cos\alpha + \sin\alpha) + 2|a_0| - \tau(|a_0| + a_0) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| \right].$$

The aim of this paper is to generalize some of the above-mentioned results. More precisely, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \geq 1$, $k_2 \geq 1$, $0<\tau_1 \leq 1$, $0<\tau_2 \leq 1$,either $k_1 a_n \geq a_{n-2} \geq ...... \geq a_3 \geq \tau_1 a_1$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_2 \geq \tau_2 a_0$,if n is odd , or $k_1 a_n \geq a_{n-2} \geq ...... \geq a_2 \geq \tau_1 a_0$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_3 \geq \tau_2 a_1$, if n is even. Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})-\tau_1(|a_1|+a_1)-\tau_2(|a_0|+a_0)+2|a_1|+|a_0|}\text{, if n is odd,}$$

and in

$$|z| < \frac{|a_0|}{k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})-\tau_1(|a_0|+a_0)-\tau_2(|a_1|+a_1)+2|a_1|+|a_0|}\text{,if n is even.}$$

Remark 1: Taking $k_1=1$, $k_2=1$, $\tau_1=1$, $\tau_2=1$, Theorem 1 reduces to Theorem B.

If z is a zero of $P(z)=\sum_{j=0}^{n} a_j z^j$, then

$$|z| = \left| z+\frac{a_{n-1}}{a_n}-\frac{a_{n-1}}{a_n} \right| \leq \left| z+\frac{a_{n-1}}{a_n} \right| + \left| \frac{a_{n-1}}{a_n} \right|.$$

Therefore, combining Theorem E and Theorem 1, we arrive at the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \geq 1$, $k_2 \geq 1$, $0<\tau_1 \leq 1$, $0<\tau_2 \leq 1$,either $k_1 a_n \geq a_{n-2} \geq ...... \geq a_3 \geq \tau_1 a_1$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_2 \geq \tau_2 a_0$,if n is odd , or $k_1 a_n \geq a_{n-2} \geq ...... \geq a_2 \geq \tau_1 a_0$ and $k_2 a_{n-1} \geq a_{n-3} \geq ...... \geq a_3 \geq \tau_2 a_1$, if n is even. Then for odd n, P(z) has all its zeros in

$$\begin{aligned}&\frac{|a_0|}{k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})-\tau_1(|a_1|+a_1)-\tau_2(|a_0|+a_0)+2|a_1|+|a_0|}\\ &\leq |z|\\ &\leq \frac{1}{|a_n|}\Big[\ k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})+2(|a_1|+|a_0|)-(|a_n|+|a_{n-1}|)-\tau_1(a_1+|a_1|)\\ &\quad -\tau_2(a_0+|a_0|)+|a_{n-1}|\Big],\end{aligned}$$

and for even n, P(z) has all its zeros in

$$\begin{aligned}&\frac{|a_0|}{k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})-\tau_1(|a_0|+a_0)-\tau_2(|a_1|+a_1)+2|a_1|+|a_0|}\\ &\leq |z|\\ &\leq \frac{1}{|a_n|}\Big[\ k_1(|a_n|+a_n)+k_2(|a_{n-1}|+a_{n-1})+2(|a_1|+|a_0|)-(|a_n|+|a_{n-1}|)-\tau_1(a_0+|a_0|)\\ &\quad -\tau_2(a_1+|a_1|)+|a_{n-1}|\Big].\end{aligned}$$

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0.$$

Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + k(|\alpha_n| + \alpha_n) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_n| + |\beta_0| + 2\sum_{j=1}^{n-1}|\beta_j|}.$$

Remark 2: If in Theorem 2, we have, in addition,

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0,$$

then $\sum_{j=1}^{n}|\beta_j - \beta_{j-}| = \beta_n - \beta_0$, and we have the following result:

Theorem 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, j=0,1,2,……,n, such that

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0.$$

Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + k(|\alpha_n| + \alpha_n) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + \beta_n - \beta_0}.$$

Remark 3: Taking $\tau = 1$, Theorem 3 reduces to Theorem C.

Combining Theorem 2 and Theorem F, we get the following result:

Corollary 2: : Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, j=0,1,2,……,n, such that

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0.$$

Then P(z) has all its zeros in

$$\frac{|a_0|}{|a_n| + k(|\alpha_n| + \alpha_n) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_n| + |\beta_0| + 2\sum_{j=1}^{n-1}|\beta_j|}$$

$$\leq |z|$$

$$\leq \frac{1}{|a_n|}[k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + |\beta_n| + 2\sum_{j=0}^{n-1}|\beta_j| + (k-1)|a_{n-1}|]$$

Theorem 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \text{ j=0,1,……,n,}$$

and for some $k \geq 1,\ 0 < \tau \leq 1$

$$k|a_n| \geq |a_{n-1}| \geq ......\geq |a_1| \geq \tau|a_0|.$$

Then P(z) does not vanish in

$$|z| < \frac{|a_0|}{k|a_n|(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|}.$$

Remark 4: Taking $\tau = 1$, Theorem 4 reduces to Theorem D.

Combining Theorem 4 and Theorem G, we get the following result:

Corollary 3 : Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some β,

$$\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}, \text{ j=0,1,……,n,}$$

and for some $k \ge 1,\ 0 < \tau \le 1$

$$k\left|a_n\right| \ge \left|a_{n-1}\right| \ge \ldots\ldots \ge \left|a_1\right| \ge \tau\left|a_0\right|.$$

Then P(z) has all its zeros in

$$\frac{\left|a_0\right|}{k\left|a_n\right|(\cos\alpha + \sin\alpha + 1) - \tau\left|a_0\right|(\cos\alpha - \sin\alpha + 1) + \left|a_0\right| + 2\sin\alpha\sum_{j=1}^{n-1}\left|a_j\right|}$$

$$\le \left|z\right|$$

$$\le \frac{1}{\left|a_n\right|}\left[k\left|a_n\right|(\cos\alpha + \sin\alpha) + 2\left|a_0\right| - \tau(\left|a_0\right| + a_0) + 2\sin\alpha\sum_{j=1}^{n-1}\left|a_j\right| + (k-1)\left|a_n\right|\right].$$

II. LEMMA

For the proofs of the above results, we need the following lemma:

Lemma: For any two complex numbers b_0 and b_1 such that $\left|b_0\right| \ge \left|b_1\right|$ and

$\left|\arg b_j - \beta\right| \le \alpha \le \frac{\pi}{2},\ j = 0,1$ for some real β,

$$\left|b_0 - b_1\right| \le (\left|b_0\right| - \left|b_1\right|)\cos\alpha + (\left|b_0\right| + \left|b_1\right|)\sin\alpha.$$

The above lemma is due to Govil and Rahman [2].

III. PROOFS OF THE THEOREMS

Proof of Theorem 1: Let n be odd. Consider the polynomial

$$\begin{aligned}
F(z) &= (1 - z^2)P(z) \\
&= (1 - z^2)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + \ldots\ldots \\
&\quad + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0 \\
&= a_0 + q(z),
\end{aligned}$$

where

$$\begin{aligned}
q(z) &= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + \ldots\ldots \\
&\quad + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (k_1 a_n - a_{n-2})z^n - (k_1 - 1)a_n z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} \\
&\quad - (k_2 - 1)a_{n-1}z^{n-1} + \ldots\ldots + (a_3 - \tau_1 a_1)z^3 + (\tau_1 - 1)a_1 z^3 + (a_2 - \tau_2 a_0)z^2 \\
&\quad + (\tau_2 - 1)a_0 z^2 + a_1 z
\end{aligned}$$

For $\left|z\right| < 1$,

$$\begin{aligned}
\left|q(z)\right| &\le \left|a_n\right| + \left|a_{n-1}\right| + k_1 a_n - a_{n-1} + (k_1 - 1)\left|a_n\right| + k_2 a_{n-1} - a_{n-3} + (k_2 - 1)\left|a_{n-1}\right| + \ldots\ldots \\
&\quad + a_3 - \tau_1 a_1 + (1 - \tau_1)\left|a_1\right| + a_2 - \tau_2 a_0 + (1 - \tau_2)\left|a_0\right| + \left|a_1\right| \\
&= k_1(\left|a_n\right| + a_n) + k_2(\left|a_{n-1}\right| + a_{n-1}) - \tau_1(\left|a_1\right| + a_1) - \tau_2(\left|a_0\right| + a_0) + 2\left|a_1\right| + \left|a_0\right| \\
&= M_1.
\end{aligned}$$

Since q(z) is analytic for $|z|<1$ and q(0)=0, it follows , by Rouche's theorem, that

$$|q(z)| \le M_1|z| \text{ for } |z|<1.$$

Hence, it follows that, for $|z|<1$,

$$\begin{aligned}|F(z)| &= |a_0 + q(z)| \\ &\ge |a_0| - |q(z)| \\ &\ge |a_0| - M_1|z| \\ &> 0\end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_1}.$$

It is easy to see that $M_1 \ge |a_0|$ and the proof is complete if n is odd.

The proof for even n is similar.

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots \\ &\quad + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots..(\alpha_2 - \alpha_1)z^2 \\ &\quad + (\alpha_1 - \alpha_0)z + a_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j \\ &= a_0 + q(z),\end{aligned}$$

where

$$\begin{aligned}q(z) &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots.. + (\alpha_2 - \alpha_1)z^2. \\ &\quad + (\alpha_1 - \alpha_0)z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j \\ &= -a_n z^{n+1} + (k\alpha_n - \alpha_{n-1})z^n - (k-1)\alpha_n z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\ &\quad + \ldots\ldots + (\alpha_2 - \alpha_1)z^2 + (\alpha_1 - \tau\alpha_0)z + (\tau - 1)\alpha_0 z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j\end{aligned}$$

For $|z|<1$,

$$\begin{aligned}|q(z)| &\le |a_n| + k\alpha_n - \alpha_{n-1} + (k-1)|\alpha_n| + \alpha_{n-1} - \alpha_{n-2} + \ldots\ldots + \alpha_2 - \alpha_1 \\ &\quad + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|) \\ &= |a_n| + k(|\alpha_n| + \alpha_n) - |\alpha_n| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j| \\ &= M_2\end{aligned}$$

Since q(z) is analytic for $|z|<1$ and q(0)=0, it follows , by Rouche's theorem, that

$$|q(z)| \le M_2|z| \text{ for } |z|<1.$$

Hence, it follows that, for $|z|<1$,

$$\begin{aligned}|F(z)| &= |a_0 + q(z)| \\ &\geq |a_0| - |q(z)| \\ &\geq |a_0| - M_2|z| \\ &> 0\end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_2}.$$

It is easy to see that $M_2 \geq |a_0|$ and the proof is complete

Proof of Theorem 4: Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots \\ &\quad + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \\ &= a_0 + q(z),\end{aligned}$$

where

$$\begin{aligned}q(z) &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots.. + (a_2 - a_1)z^2 \\ &\quad + (a_1 - a_0)z \\ &= -a_n z^{n+1} + (ka_n - a_{n-1})z^n - (k-1)a_n z^n + (a_{n-1} - a_{n-2})z^{n-1} \\ &\quad + \ldots\ldots + (a_2 - a_1)z^2 + (a_1 - \tau a_0)z + (\tau - 1)a_0 z.\end{aligned}$$

For $|z| < 1$, we have , by using the lemma,

$$\begin{aligned}|q(z)| &\leq |a_n| + |ka_n - a_{n-1}| + (k-1)|a_n| + |a_{n-1} - a_{n-2}| + \ldots\ldots + |a_2 - a_1| \\ &\quad + |a_1 - \tau a_0| + (1-\tau)|a_0| \\ &\leq |a_n| + (k|a_n| - |a_{n-1}|)\cos\alpha + (k|a_n| + |a_{n-1}|)\sin\alpha + (k-1)|a_n| \\ &\quad + (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots\ldots \\ &\quad + (|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha + (|a_1| - \tau|a_0|)\cos\alpha \\ &\quad + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0| \\ &= k|a_n|(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| \\ &= M_3\end{aligned}$$

Since q(z) is analytic for $|z| < 1$ and q(0)=0, it follows , by Rouche's theorem, that

$$|q(z)| \leq M_3|z| \text{ for } |z| < 1.$$

Hence, it follows that, for $|z| < 1$,

$$\begin{aligned}|F(z)| &= |a_0 + q(z)| \\ &\geq |a_0| - |q(z)| \\ &\geq |a_0| - M_3|z| \\ &> 0\end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_3}.$$

It is easy to see that $M_3 \geq |a_0|$ and the proof is complete

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