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On The Degree of Approximation of Functions by the Generalized Polynomials

Anwar Habib[1], Adel S. Bahakeem[2]
Jubail Industrial College ,Jubail Industrial city 31961, K S A

Abstract:- Popoviciu(1935) proved his result for Bernstein Polynomials. We have tested the degree of approximation of function by a newly defined Generalized Polynomials, and so the corresponding results of Popoviciuhave been extended for Lebesgue integrable function in L_1-norm by our newly defined Generalized Polynomials

$$U_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha)$$

where

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

Keywords:-Bernstein Polynomials, Degree of Approximation, GeneralizedPolynomial, L_1 -norm,LebesgueIntegrable function.

I. INTRODUCTION AND RESULTS

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x) \ldots\ldots(1.1)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \ldots\ldots\ldots.(1.2)$$

Bernstein (1912-13) proved that if $f(x)$ is continuous in closed interval [0,1], then $B_n^f(x)$ tends to $f(x)$uniformly as $n \to \infty$. This Yields a simple constructive proof of Weierstrass's approximation theorem.

A more precise version of this result due to popoviciu(1935) states that

$$\left|B_n^f(x) - f(x)\right| \le \frac{5}{2} w_f(n^{-1/2})$$

wherew_f is the uniform modulus of continuity of f defined by

$$w_f(h) = \max\{|f(x) - f(h)|; x, y \in [0,1], |x-y| \le h\}$$

A slight modification of Bernstein polynomials due to Kantorovitch [5] makes it possible to approximate Lebesgueintegrable function in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x) \ldots(1.3)$$

where$p_{n,k}(x)$ is defined by (1.2)

By Abel's formula ([4])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1} y(y+(n-k)\alpha)^{n-k-1} \ldots.(1.4)$$

If we put= $1-x$, we obtain([3])

$$1 = \sum_{k=0}^{n}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \ldots..(1.5)$$

Thus defining

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \ldots..(1.6)$$

we have

$$\sum_{k=0}^{n} q_{n,k}(x;\alpha) = 1 \ldots.(1.7)$$

We now define apolynomial([2])analogous to (1.3)

$$U_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha) \ldots(1.8)$$

where $q_{n,k}(x;\alpha)$ is defined in(1.6) and moreover when $\alpha = 0$,(1.6) and (1.8) reduces to (1.2) and (1.3) respectively.
In this paper, we shall test the degree of approximation by our polynomial (1.8) for Lebesgueintegrable function in L_1-norm.
In fact we state our results as follows
Theorem 1: If $f(x)$ is continuousLebesgueIntegrable function on [0,1] and w(δ) is the modulus of continuity of $f(x)$, then for $\alpha = \alpha_n = o(1/n)$ we have

$$\left|U_n^{\alpha}(f,x) \quad - f(x)\right| \le \frac{3}{2} w\left(\frac{1}{\sqrt{n}}\right)$$

Theorem 2: If $f(x)$ is continuous Lebesgueintegrable function on [0,1] such that its first derivative is bounded and $w_1(\delta)$ is the modulus of continuity of $f\acute{(x)}$, for $\alpha = \alpha_n = o(1/n)$ we have

$$\left|U_n^{\alpha}(f,x) \quad - f(x)\right| \le \frac{3}{4}\frac{1}{\sqrt{n}} w\left(\frac{1}{\sqrt{n}}\right) + o\left(\frac{1}{n}\right) .$$

II. LEMMAS

In order to prove our results we need the following lemmas [2]
Lemma 2.1: For all value of x

$$\sum_{k=0}^{n} k q_{n,k}(x;\alpha) \le \frac{1+n\alpha}{1+\alpha} nx - \frac{n(n-1)x\alpha}{1+2\alpha} .$$

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n} k(k-1) q_{n,k}(x;\alpha) \le n(n-1)[(x+2\alpha)\{\frac{1+n\alpha}{(1+2\alpha)^2} - \frac{(n-2)\alpha}{(1+3\alpha)^2}\}$$
$$+(n-2)\alpha^2\{\frac{1+n\alpha}{(1+3\alpha)^3} - \frac{(n-3)\alpha}{(1+4\alpha)^3}\}] .$$

Lemma 2.3: For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2 dt\right\} q_{n,k}(x;\alpha) \le \frac{x(1-x)}{n} .$$

III. PROOF OF THE THEOREMS

Proof of theorem 1:

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \le (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |f(t)-f(x)| dt\right\} q_{n,k}(x;\alpha)$$

Using the property of modulus of continuity

$$|f(x_2) - f(x_1)| \le w(|x_2 - x_1|)$$

$$w(\lambda\delta) \le ([\lambda]+1)w(\delta) \le (\lambda+1)w(\delta) , \ \lambda > 0 \quad \text{--------(3.1)}$$

we obtain

$$|f(x) - f(t)| \le w(|x-t|)$$
$$= w\left(\frac{1}{\delta}|x-t|\delta\right)$$
$$= \left(1 + \frac{1}{\delta}|x-t|\right) w(\delta)$$

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \le (n+1) w(\delta) \sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} \left(1 + \frac{1}{\delta}|x-t|\right) dt\right\} q_{n,k}(x;\alpha)$$
$$= w(\delta)[(n+1)\delta^{-1} \sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |x-t| dt\right\} q_{n,k}(x;\alpha) \quad \text{------} \quad (3.2)$$

then by Cauchy ‘s inequality , we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |x-t|dt\right\} q_{n,k}(x;\alpha)$$

$$\leq [(n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (x-t)^2 dt\right\} q_{n,k}(x;\alpha)]^{1/2}$$

By lemma 3 and the fact $x(1-x) \leq \frac{1}{4} on$ [0,1]

$$\leq \left(\frac{1}{4n}\right)^{1/2} \text{-------------------------------(3.3)}$$

and hence from (3.2) and (3.3) we have

$$\left|U_n^{\alpha}(f,x) - f(x)\right| \leq [1 + \delta^{-1}\left(\frac{1}{2\sqrt{n}}\right)]w(\delta)$$

But for $\delta = n^{-1/2}$

$$\left|U_n^{\alpha}(f,x) \quad - f(x)\right| \leq \frac{3}{2} w\left(\frac{1}{\sqrt{n}}\right)$$

which completes the proof of theorem 1.

Proof of theorem 2:

By applying the Mean Value Theorem of differential calculus, we can write

$$f(x) - f(t) = (x-t)\ f'(\xi)$$

$$= (x-t)\ f'(x) + (x-t)\{\ f'(\xi) - \ f'(x)\} \text{--------(3.4)}$$

where ξ is an interior point of the interval determined by x and t .

If we multiply (3.4) by $(n+1)q_{n,k}(x;\alpha)$ and integrate it from $\frac{k}{n+1}$ to $\frac{k+1}{n+1}$ and sum over k , there follows

$$f(x) - U_n^{\alpha}(f,x)$$

$$= (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} |f(x) - f(t)|dt\right\} q_{n,k}(x;\alpha)$$

$$= (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (x-t)\ f'(x)\, dt\right\} q_{n,k}(x;\alpha)$$

$$+ (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (x-t)\{\ f'(\xi) - \ f'(x)\}dt\right\} q_{n,k}(x;\alpha)$$

$$\left|(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (x-t)\ f'(x)\, dt\right\} q_{n,k}(x;\alpha)\right|$$

$$= \left|(n+1)\sum_{k=0}^{n}\left\{\frac{x}{n+1} - \frac{2k+1}{2(n+1)^2}\right\}\ f'(x)\ q_{n,k}(x;\alpha)\right|$$

$$= \left|\sum_{k=0}^{n}\left\{x - \frac{k}{n+1} - \frac{1}{2(n+1)}\right\}\ f'(x)\ q_{n,k}(x;\alpha)\right|$$

By Lemma1 and (1.7) , we have

$$= \left|\left\{x - \frac{1}{n+1}\left[\frac{1+n\alpha}{1+\alpha} nx - \frac{n(n-1)x\alpha}{1+2\alpha}\right] - \frac{1}{2(n+1)}\right\}\ f'(x)\right|$$

$\leq \frac{M}{n}$where as $\left|\ f'(x)\right| \leq M$ and $\alpha = \alpha_n = o(1/n)$ and for large nand by (3.1)

$$\left|\ f'(\xi) - \ f'(x)\right| \leq w_1\left(\left|\xi - x\right|\right) \leq \left(1 + \frac{1}{\delta}\left|\xi - x\right|\right) w_1(\delta)$$

where$\delta$is a positive number does not defined on k .

Consequently we can have

$$\left|f(x) - U_n^{\alpha}(f,x)\right| \leq \frac{M}{n} + w_1(\delta)[(n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |x-t|dt\right\} q_{n,k}(x;\alpha)$$

$$+\frac{1}{\delta}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (x-t)^2 dt\right\} q_{n,k}(x;\alpha)]$$

Hence by (3.3) and the Lemm3 and with the fact $x(1-x) \leq \frac{1}{4} on$ [0,1]

We have

$\left|f(x) - U_n^{\alpha}(f,x)\right| \le \frac{M}{n} + w_1(\delta)\{\frac{1}{2\sqrt{n}} + \frac{1}{\delta}\left(\frac{1}{4n}\right)$

But for $\delta = n^{-1/2}$

$$\le \frac{M}{n} + w_1\left(\frac{1}{\sqrt{n}}\right)\{\frac{1}{2\sqrt{n}} + \frac{1}{4\sqrt{n}}$$

$$\le \frac{3}{4\sqrt{n}} w_1\left(\frac{1}{\sqrt{n}}\right) + o(\frac{1}{n})$$

which completes the proof of theorem 2.

IV. CONCLUSION

The results of Popoviciuhave been extended for LebesgueIntegrable function in L_1-norm by our newly defined Generalized Polynomials.

REFERENCES

[1]. Anwar Habib (1981).“On the degree of approximation of functions by certain new Bernstein typePolynomials”. Indian J. pure Math.,**12**(7):882-888.

[2]. Anwar Habib & Saleh Al Shehri(2012) “On Generalized Polynomials I ‘ International Journal of Engineering Research and Development e-ISSN:2278-067X, 2278-800X, Volume 5, Issue4 ,December **2012** , pp.18-26

[3]. Cheney, E.W. , and Sharma, A.(1964).”On a generalization of Bernstein polynomials”.Rev. Mat. Univ. Parma(2),**5**,77-84.

[4]. Jensen, J. L. W. A. (1902). “Sur uneidentité Abel et surd’autressformulesamalogues”.Acta Math. , **26**, 307-18

[5]. Kantorovitch, L.V.(1930).” Sur certainsdéveloppmentssuivantléspôlynômesdé la formeS.Bernstein I,II.” C.R. Acad. Sci. URSS,**20**,563-68,595-600.

[6]. Lorentz, G.G. (1955). “Bernstein Polynomials”. University of Toronto Press, Toronto .

[7]. Popoviciu (1935) .Sur l’approximation des fonctionscovexesd’ordre superior. Mathematica(cluj) 10 (1935), 49-54 .