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On The Zeros of a Polynomial in a Given Circle

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Abstract:- In this paper we discuss the problem of finding the number of zeros of a polynomial in a given circle when the coefficients of the polynomial or their real or imaginary parts are restricted to certain conditions. Our results in this direction generalize some well- known results in the theory of the distribution of zeros of polynomials.
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I. INTRODUCTION AND STATEMENT OF RESULTS

In the literature many results have been proved on the number of zeros of a polynomial in a given circle. In this direction Q. G. Mohammad [6] has proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0} .$$

K. K. Dewan [2] generalized Theorem A to polynomials with complex coefficients and proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 > 0 .$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|} .$$

Theorem C: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β ,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......n$$

and

$$|a_n| \ge |a_{n-1}| \ge ..... \ge |a_1| \ge |a_0| .$$

Then the number f zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}\quad.$$

The above results were further generalized by researchers in various ways.
M. H. Gulzar[4,5,6] proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_\lambda,k\alpha_\lambda\ge\alpha_{\lambda-1}\ge.......\ge\alpha_1\ge\tau\alpha_0,$$

for $k\ge 1,0<\tau\le 1,0\le\lambda\le n.$ Then the number of zeros of P(z) in $|z|\le\delta,0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}\quad.$$

Theorem E: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_1\ge\tau\alpha_0,$$

for some $\rho\ge 0,0<\tau\le 1,$,then the number f zeros of P(z) in $|z|\le\delta,0<\delta<1,$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Theorem F: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,......n$$

and

$$|\rho+a_n|\ge|a_{n-1}|\ge.....\ge|a_1|\ge\tau|a_0|\ ,$$

for some $\rho\ge 0,$,then the number of zeros of P(z) in $|z|\le\delta,0<\delta<1,$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-|a_0|(\cos\alpha-\sin\alpha-1)}{|a_0|}\quad.$$

The aim of this paper is to find a bound for the number of zeros of P(z) in a circle of radius not necessarily less than 1. In fact , we are going to prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial of degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_\lambda,k\alpha_\lambda\ge\alpha_{\lambda-1}\ge.......\ge\alpha_1\ge\tau\alpha_0,$$

for $k \geq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n.$ Then the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1.$

Remark 1: Taking R=1 and $c = \frac{1}{\delta}$ in Theorem 1, it reduces to Theorem D .

If the coefficients a_j are real i.e. $\beta_j = 0, \forall j$, then we get the following result from Theorem 1:

Corollary 1: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_\lambda, ka_\lambda \geq a_{\lambda-1} \geq .... \geq a_1 \geq \tau a_0,$$

for $k \geq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n.$ Then the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|a_n| + a_n + (k-1)(|a_\lambda| + a_\lambda) + 2|a_0| - \tau(|a_0| + a_0)]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[|a_n| + a_n + (k-1)(|a_\lambda| + a_\lambda) + |a_0| - \tau(|a_0| + a_0)]}{|a_0|}$$

for $R \leq 1.$

Applying Theorem 1 to the polynomial –iP(z) , we get the following result:

Theorem 2: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_\lambda, k\beta_\lambda \geq \beta_{\lambda-1} \geq .... \geq \beta_1 \geq \tau\beta_0,$$

for $k \geq 1, 0 < \tau \leq 1, 0 \leq \lambda \leq n.$ Then the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|\beta_n| + \beta_n + (k-1)(|\beta_\lambda| + \beta_\lambda) + 2|\beta_0| - \tau(|\beta_0| + \beta_0) + 2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[|\beta_n| + \beta_n + (k-1)(|\beta_\lambda| + \beta_\lambda) + |\beta_0| + |\alpha_0| - \tau(|\beta_0| + \beta_0) + 2\sum_{j=1}^{n}|\alpha_j|]}{|a_0|}$$

for $R \le 1$.

Taking $\lambda = n$ in Theorem 1, we get the following result :

Corollary 2: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$k\alpha_n \ge \alpha_{n-1} \ge .... \ge \alpha_1 \ge \tau\alpha_0,$$

for $k \ge 1, 0 < \tau \le 1, 0 \le \lambda \le n.$ Then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c} \log \frac{R^{n+1}[k(|\alpha_n| + \alpha_n) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n} |\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c} \log \frac{|a_0| + R[k(|\alpha_n| + \alpha_n) + |\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n} |\beta_j|]}{|a_0|}$$

for $R \le 1$.

Theorem 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0,$$

for some $\rho, 0 < \tau \le 1,$,then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$, does not exceed

$$\frac{1}{\log c} \log \frac{R^{n+1}[|\rho| + \rho + |\alpha_n| + \alpha_n - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log c} \log \frac{|a_0| + R[|\rho| + \rho + |\alpha_n| + \alpha_n - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|]}{|a_0|}$$

for $R \le 1$.

Remark 2: Taking R=1 and $c = \frac{1}{\delta}$ in Theorem 3, it reduces to Theorem E .

If the coefficients a_j are real i.e. $\beta_j = 0, \forall j$, then we get the following result from Theorem 3:

Corollary 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$\rho + a_n \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0,$$

for some $\rho, 0 < \tau \le 1.$ Then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$, does not exceed

$$\frac{1}{\log c} \log \frac{R^{n+1}[|\rho| + \rho + |a_n| + a_n - \tau(|a_0| + a_0) + 2|a_0|]}{|a_0|} \quad \text{for } R \ge 1$$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[|\rho|+\rho+|a_n|+a_n-\tau(|a_0|+a_0)+|a_0|]}{|a_0|}$$

Applying Theorem 3 to the polynomial –iP(z) , we get the following result:

Theorem 4: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and

$$\rho+\beta_n\geq\beta_{n-1}\geq......\geq\beta_1\geq\tau\beta_0,$$

for some $\rho\geq 0, 0<\tau\leq 1.$ Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|\rho|+\rho+|\beta_n|+\beta_n-\tau(|\beta_0|+\beta_0)+2|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[|\rho|+\rho+|\beta_n|+\beta_n-\tau(|\beta_0|+\beta_0)+|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\leq 1.$

Taking $\rho=(k-1)\alpha_n, k\geq 1$ in Corollary 3 , we get the following result:

Corollary 4: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and

$$k\alpha_n\geq\alpha_{n-1}\geq......\geq\alpha_1\geq\tau\alpha_0,$$

for some $\rho, 0<\tau\leq 1.$ Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R\leq 1.$

Theorem 5: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n such that

$$|\rho+a_n|\geq|a_{n-1}|\geq......\geq|a_1|\geq\tau|a_0|$$

for some $\rho, 0<\tau\leq 1.$ Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{(|\rho|+|a_n|)(\cos\alpha+\sin\alpha+1)-\tau|\alpha_0|(\cos\alpha-\sin\alpha+1)+2|\alpha_0|\}]$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0| + R\{(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|\}]$$

for $R \leq 1$.

For different values of the parameters k, ρ, τ in the above results, we get many other interesting results.

II. LEMMAS

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z| \leq R$, but not identically zero, f(0) $\neq$ 0 and $f(a_k) = 0, k = 1,2,......, n$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{Re}^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n}\log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic and $|f(z)| \leq M(r)$ in $|z| \leq r$, then the number of zeros of f(z) in $|z| \leq \frac{r}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β, $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, 0 \leq j \leq n,$ and $|a_j| \geq |a_{j-1}|, 0 \leq j \leq n,$ then any t>0,

$$|ta_j - a_{j-1}| \leq (t|a_j| - |a_{j-1}|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

Lemma 3 is due to Govil and Rahman [4].

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

F(z) =(1-z)P(z)

$$\begin{aligned}
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + a_0 + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} \\
&\quad + [(k\alpha_\lambda - \alpha_{\lambda-1}) - (k-1)\alpha_\lambda]z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ... \\
&\quad + [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + i\sum_{j=0}^{n}(\beta_j - \beta_{j-1})z^j
\end{aligned}$$

For $|z| \leq R$, we have by using the hypothesis

$$\begin{aligned}
|F(z)| &\leq |a_n|R^{n+1} + |a_0| + |\alpha_n - \alpha_{n-1}|R^n + ...+ |\alpha_{\lambda+1} - \alpha_\lambda|R^{\lambda+1} + |k\alpha_\lambda - \alpha_{\lambda-1}|R^\lambda \\
&\quad + (k-1)|\alpha_\lambda|R^\lambda + |\alpha_{\lambda-1} - \alpha_{\lambda-2}|R^{\lambda-1} + ...+ |\alpha_1 - \tau\alpha_0|R + (1-\tau)|\alpha_0|R \\
&\quad + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)
\end{aligned}$$

$$\leq |a_n|R^{n+1} + |a_0| + R^n[\alpha_n - \alpha_{n-1} + ...+ \alpha_{\lambda+1} - \alpha_\lambda + k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda|$$

$$+ \alpha_{\lambda-1} - \alpha_{\lambda-2} + ... + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|]$$

$$= |a_n|R^{n+1} + |a_0| + R^n[\alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n|$$

$$+ 2\sum_{j=1}^{n-1}|\beta_j|]$$

$$\leq R^{n+1}[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| \;\; + 2\sum_{j=0}^{n}|\beta_j|] \qquad \text{for } R \geq 1$$

and

$$|F(z)| \leq |a_0| + R[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|]$$

for $R \leq 1$.

Therefore , by Lemma 3, it follows that the number of zeros of F(z) and hence P(z) in $|z| \leq \frac{R}{c}\,(R > 0, c > 0)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|}{|a_0|}$$

for $R \leq 1$.

That proves Theorem 1.

Proof of Theorem 3: Consider the polynomial

F(z) =(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + a_0 - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + ... + (\alpha_2 - \alpha_1)z^2$$

$$+ [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + i\sum_{j=0}^{n}(\beta_j - \beta_{j-1})z^j \; .$$

For $|z| \leq R$, we have by using the hypothesis

$$|F(z)| \leq |a_n|R^{n+1} + |a_0| + |\rho|R^n + |\rho + \alpha_n - \alpha_{n-1}|R^n + ...+ |\alpha_2 - \alpha_1|R^2 + |\alpha_1 - \tau\alpha_0|R$$

$$+ (1-\tau)|\alpha_0|R + \sum_{j=0}^{n}(|\beta_j| + |\beta_{j-1}|)R^j$$

$$\leq R^{n+1}[|\alpha_n| + |\alpha_0| + |\rho| + \rho + \alpha_n - \alpha_{n-1} + ...+ \alpha_2 - \alpha_1 + \alpha_1 - \tau\alpha_0$$

$$+ (1-\tau)|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

$$= R^{n+1}[|\alpha_n| + \alpha_n + |\rho| + \rho - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|] \quad \text{for } R \geq 1$$

and

$$|F(z)| \leq |a_0| + R[|\rho| + \rho + |\alpha_n| + \alpha_n - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|] \qquad \text{for } R \leq 1.$$

Therefore , by Lemma 2, it follows that the number of zeros of F(z) and hence P(z) in $|z| \leq \frac{R}{c}\,(R > 0, c > 0)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[|\rho| + \rho + |\alpha_n| + \alpha_n - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[|\rho| + \rho + |\alpha_n| + \alpha_n - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1$.

That proves Theorem 3.

Proof of Theorem 5: Consider the polynomial

F(z) =(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + a_0 - \rho z^n + (\rho + a_n - a_{n-1})z^n + ... + (a_2 - a_1)z^2$$
$$+ [(a_1 - \tau a_0) + (\tau a_0 - a_0)]z\,.$$

For $|z| \leq R$, we have by using the hypothesis and Lemma 3

$$|F(z)| \leq |a_n|R^{n+1} + |a_0| + |\rho|R^n + |\rho + a_n - a_{n-1}|R^n + ...+ |a_2 - a_1|R^2 + |a_1 - \tau a_0|R$$
$$+ (1-\tau)|\alpha_0|R$$
$$\leq |a_n|R^{n+1} + |a_0| + |\rho|R^n + [(|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha]R^n$$
$$+ ......+ [(|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha]R^2 + (1-\tau)|a_0|R^2$$
$$+ [(|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha]R$$
$$\leq R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|\alpha_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

for $R \geq 1$

and

$$\leq |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|]$$

for $R \leq 1$.

Hence , by Lemma 2, it follows that the number of zeros of F(z) and therefore P(z) in $|z| \leq \frac{R}{c}\,(R > 0, c > 0)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|\alpha_0|(\cos\alpha - \sin\alpha + 1) + 2|\alpha_0|\}]$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0| + R\{(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|\}]$$

for $R \leq 1$.

That completes the proof of Theorem 5.

REFERENCES

[1]. L.V.Ahlfors, Complex Analysis, 3rd edition, Mc-Grawhill

[2]. K. K. Dewan, Extremal Properties and Coefficient estimates for Polynomials with restricted Zeros and on location of Zeros of Polynomials, Ph.D. Thesis, IIT Delhi, 1980.

[3]. N. K. Govil and Q. I. Rahman, On the Enestrom- Kakeya Theorem, Tohoku Math. J. 20(1968),126-136.

[4]. M. H. Gulzar, On the Zeros of a Polynomial inside the Unit Disk, IOSR Journal of Engineering, Vol.3, Issue 1(Jan.2013) IIV3II, 50-59.

[5]. M. H. Gulzar, On the Number of Zeros of a Polynomial in a Prescribed Region, Research Journal of Pure Algebra-2(2), 2012, 35-49.

[6]. M. H. Gulzar, On the Zeros of a Polynomial, International Journal of Scientific and Research Publications, Vol. 2, Issue 7,July 2012,1-7.

[7]. Q. G. Mohammad, On the Zeros of Polynomials, Amer. Math. Monthly 72(1965), 631-633.